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Some aspects of the analysis of Einstein equations

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Chapter 1

Introduction

In this habilitation thesis, I will present the works I have done since my PHD. They are all related to the study of Einstein equations so I start with a general introduction on the topic, and more precisely on the Cauchy problem in general relativity. In the last part of the introduction, I will present briefly the results I have obtained since my PHD. Some of them will be explained in more details in the other chapters.

1.1 Einstein equations

In general relativity, the space-time is described by $(\mathcal{M}, g)$ where $\mathcal{M}$ is a manifold, usually 4 dimensional, but not always, and g is a Lorentzian metric. Bodies which are in free fall follow the geodesic in the metric g, so the metric encodes the phenomenon of gravitation. The geometry is linked to the densities of matter and fields present in the universe by the celebrated Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \tag{1.1.1}$$

In these equations $R_{\mu\nu}$ is the Ricci tensor : it is a contraction of the Riemann curvature tensor. The scalar R is the trace of the Ricci tensor, also called the scalar curvature. The tensor $T_{\mu\nu}$ is the stress-energy tensor : its form depends on the matter model under study. The contracted Bianchi equations, which are the algebraic relations $D^\mu(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$ imply that $D^\mu T_{\mu\nu} = 0$. These are local conservation laws for the fields : these equations depend on the metric g, making (1.1.1) a coupled system of equations for the metric g and the fields.

A manifold of dimension n is locally diffeomorphic to $\mathbb{R}^n$. This means that in the neighbourhood of any point, we can introduce a coordinate system x^μ, and write the metric g in the form $g_{\mu\nu} dx^\mu dx^\nu$. The Riemann tensor, the Ricci tensor and the scalar curvature can be then computed explicitly from the metric coefficients $g_{\mu\nu}$. At this stage of the introduction, the exact formula is not very relevant, but we can say that they involve only expressions of the form $g\partial^2 g + \partial g \partial g$. Therefore, Einstein equations consist of second order non linear equations for the metric coefficients.

1.1.1 Examples of stress-energy tensors

As stated before, the form of the stress-energy tensor depends on the field we are considering. We give here a selection of examples which will appear at some point in this manuscript.

Scalar field: described by a function ϕ, with possibly a potential $V(\phi)$: the stress-energy tensor is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi - g_{\mu\nu} V(\phi).$$

The equations $D^\mu T_{\mu\nu} = 0$ yield the wave equation $\Box_g \phi - V'(\phi) = 0$.

Perfect fluid: described by a timelike vector field u, with energy density ρ and pressure p. The stress-energy tensor is

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu).$$

The equations $D^\mu T_{\mu\nu} = 0$ yield Euler equations in curved space-times.

Vlasov field: described by a density f in position and frequency. The stress-energy tensor is given by

$$T_{\mu\nu} = \int_{P_x^*} f(x,p) p_\mu p_\nu d\sigma_{P_x^*},$$

where $P_x^* = \{p \in T_x^* \mathcal{M},\ g^{\mu\nu} p_\mu p_\nu = 0\}$ is called the mass shell, and $d\sigma_{P_x^*}$ is the measure induced by the metric g on the mass shell. The equations $D^\mu T_{\mu\nu} = 0$ yield the transport equation along geodesics for f

$$g^{\alpha\beta} p_\beta \partial_{x^\alpha} f - \frac{1}{2} \partial_{x^i} g^{\alpha\beta} p_\alpha p_\beta \partial_{p_i} f = 0.$$

Electromagnetic field: described by an antisymmetric closed 2 form $F_{\alpha\beta}$. The closeness assumption can be written $D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0$. The stress-energy tensor is

$$T_{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$

The equations $D^\mu T_{\mu\nu} = 0$ yield $D^\mu F_{\mu\nu} = 0$.

Yang-Mills field: described by a potential A which is a 1-form with value in a Lie algebra $\mathcal{G}$ of $N \times N$ matrices. The potential induces a connection

$$D^{(A)} X = DX + [A, X],$$

where D is the Levi-Civita connection, and a curvature

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu + [A_\mu, A_\nu].$$

The stress-energy tensor is also

$$T_{\mu\nu} = \langle F_\mu{}^\lambda, F_{\nu\lambda} \rangle - \frac{1}{4} g_{\mu\nu} \langle F^{\alpha\beta}, F_{\alpha\beta} \rangle,$$

where $\langle , \rangle$ is the scalar product in the space of matrices. The equations $D^\mu T_{\mu\nu} = 0$ yield

$$D_\mu^{(A)} F^{\mu\nu} = 0.$$

1.1.2 Some explicit solutions in vacuum

In vacuum, that is to say with the stress-energy tensor $T_{\mu\nu}$ equals to zero, Einstein equations are equivalent to the requirement that the Ricci tensor $R_{\mu\nu}$ vanishes. A trivial solution is given by Minkowski metric

$$m = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

This metric is flat : this means that its Riemann's curvature tensor vanishes. However, in dimension greater or equal to four, the vanishing of the Ricci tensor does not imply the vanishing of the Riemann tensor, and Minkowski metric is far from being the only solution to Einstein vacuum equation.

Another well-known special solution is given by the Schwarzschild metric, which is static and spherically symmetric and is given for $r > 2m$ by the formula

$$g_S = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2$$

This solution, derived by Schwarzschild in 1916 to describe the geometry in the exterior of a star with spherical symmetry, can be extended beyond $r = 2m$ which is a coordinate singularity to produce a black-hole solution. Another family of black-holes solutions, static and axisymmetric, is also known in exact form, the so called Kerr solutions.

We finish this short section with the presentation of a family of non static but almost explicit solutions : the plane waves. A metric of the form

$$g = -dudv + B(u)^2 (e^{\omega(u)} dx^2 + e^{-\omega(u)} dy^2) \tag{1.1.2}$$

is a solution to Einstein vacuum equations if and only if

$$B''(u) - \omega(u)^2 B(u) = 0. \tag{1.1.3}$$

The coefficient $\omega(u)$ is a function of u which can be chosen arbitrarily. These solutions are the simplest toy models to understand gravitational waves : the deformation of the space-time occurs in the x and y direction, which are orthogonal to the direction of propagation u.

1.2 Cauchy problem

As already guessed with the presentation of the plane wave solutions, Einstein equations in vacuum are dynamical, and they can be formulated as a Cauchy problem. We will present it here in the vacuum case, but there are also similar formulations and results in the non vacuum case.

The initial data for Einstein equations are given by a triplet $(\Sigma, \bar{g}, K)$, where $(\Sigma, \bar{g})$ is a Riemannian manifold, and K is a symmetric two tensor on Σ. Solving Einstein equations with initial data $(\Sigma, \bar{g}, K)$ consists in finding a Lorentzian manifold $(\mathcal{M}, g)$, solution of Einstein equations such that

- Σ can be embedded in $\mathcal{M}$,
- The Riemannian metric $\bar{g}$ is the pull-back of the restriction of the Lorentzian metric g to the embedded Σ.

- K is the second fundamental form of the embedding of Σ in $\mathcal{M}$. In PDE terms, it is similar to the data of $\partial_t \phi$ at time zero when we solve a wave equation.

A space-time $(\mathcal{M}, g)$ satisfying these conditions is called a development of $(\Sigma, \bar{g}, K)$

The initial data cannot be prescribed arbitrarily. Indeed, doing a $n+1$ decomposition of the metric and the Einstein tensor, one can compute that the equations $R_{00} - \frac{1}{2} R g_{00} = 0$ and $R_{0i} = 0$, where 0 is the time index and i the space indices, depend only on the initial data. The corresponding equations, called the constraint equations, are

$$\begin{aligned} \bar{R} + \tau^2 - |K|^2_{\bar{g}} &= 0 \\ \nabla^i_{\bar{g}} K_{ij} - \partial_j \tau &= 0, \end{aligned} \tag{1.2.1}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$, τ is the trace of K with respect to the metric $\bar{g} : \tau = \bar{g}^{ij} K_{ij}$ and $\nabla_{\bar{g}}$ is the Levi-Civita connection associated to $\bar{g}$. These equations form an under-determined set of equations.

In their seminal work, Choquet-Bruhat and Geroch proved the local well posedness of Einstein equations

Theorem 1.2.1. *[23], [12] Let $(\Sigma, \bar{g}, K)$ be a set of sufficiently regular initial data, satisfying the constraint equation. Up to diffeomorphism, there exists a unique maximal, globally hyperbolic development $(\mathcal{M}, g)$ of $(\Sigma, \bar{g}, K)$ solution of Einstein equations.*

1.2.1 The constraint equations

Studying the solutions to the constraint equations (1.2.1) is a field of study in itself. Here, we will just give some ideas of the most used method for their resolution, the conformal method. The equations (1.2.1) form an underdetermined system of equations for $\bar{g}$ and K. A way to parametrize the solutions, introduced for the first time by Lichnerowitz in [56], consists in looking for solution $\hat{g}, \hat{K}$ of (1.2.1) on Σ of dimension n, of the form

$$\hat{g} = \varphi^{\frac{4}{n-2}} \gamma, \qquad \hat{K} = \varphi^{-2\frac{n+2}{n-2}} ((\mathcal{L}_\gamma W)^{ij} + U^{ij}) + \frac{1}{n} \hat{g}^{ij} \tau,$$

where $\mathcal{L}_\gamma$ is the conformal Killing operator

$$(\mathcal{L}_\gamma W)_{\mu\nu} = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{n} \nabla^\beta W_\beta \gamma_{ij}, \quad \forall\, \mu, \nu \in \overline{1, n},$$

where ∇ is the Levi-Civita connection associated to γ, and U a divergenceless, traceless $2-$tensor. The quantities γ, U and τ should be seen as parameters. Then the constraint equations give a system of determined elliptic equations for φ, W which can be written

$$\begin{aligned} \frac{4(n-1)}{n-2} \Delta_\gamma \varphi + R_\gamma \varphi &= -\frac{n-1}{n} \tau^2 \varphi^{\frac{n+2}{n-2}} + \frac{|U + \mathcal{L}_\gamma W|^2_g}{\varphi^{\frac{3n-2}{n-2}}} \\ \overrightarrow{\Delta}_\gamma W &= \frac{n-1}{n} \varphi^{\frac{4}{n-2}} d\tau, \end{aligned}$$

where $\overrightarrow{\Delta}_\gamma W = \nabla^\mu (\mathcal{L}_\gamma W)_{\mu\nu}$ is the conformal vectorial Laplacian. There is a vast literature describing the resolution of this system, on compact, asymptotically flat or asymptotically hyperbolic manifolds, depending on whether the mean curvature τ is constant, in which case the system decouples, is close, or is far from being constant (see [18] for a survey on the subject).

1.2.2 Wave coordinates

Solving Einstein equations requires some gauge choices. In any coordinate chart (x^α), the Ricci tensor can be computed

$$R_{\nu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + \frac{1}{2}P_{\mu\nu}(g)(\partial g, \partial g),$$

where

$$H^\alpha = \partial_\beta g^{\beta\alpha} - \frac{1}{2}g^{\beta\rho}\partial_\alpha g_{\nu\rho}, \tag{1.2.2}$$

and $P_{\mu\nu}(g)(\partial g, \partial g)$ is a quadratic form in the first derivatives of g. In fact, one can compute that $H^\alpha = \Box_g x^\alpha$: this is the d'Alembertian in the metric g of the coordinate function x^α. Assuming that $H^\alpha = 0$ is a condition on the coordinates, which is the analogue of the Lorentz gauge for Maxwell equations, and allows to recast Einstein equations in vacuum as a system of quasilinear wave equations for the metric coefficients

$$\Box_g g_{\mu\nu} = P_{\mu\nu}(g)(\partial g, \partial g). \tag{1.2.3}$$

Let us sketch the strategy to solve Einstein equations in wave coordinates, which is important to understand even if we work in other gauges since there are similar aspects in the philosophy. We recall that the initial data for Einstein equations are $(\Sigma, \bar{g}, K)$, with $\bar{g}$ a Riemannian metric on Σ and K a symmetric 2-tensor, solution to the constraint equations.

- To solve (1.2.3) we need the initial data for $g_{\mu\nu}$ and $\partial_t g_{\mu\nu}$ at time $t = 0$. We choose them following the procedure.
  - We take $g_{ij} = \bar{g}_{ij}$,
  - The coefficients g_{00} and g_{0i} are free : we can choose $g_{00} = -1$ and $g_{0i} = 0$ without loss of generality,
  - We take $\partial_t g_{ij} = K_{ij}$.
  - We choose $\partial_t g_{00}$ and $\partial_t g_{0i}$ in order for the wave coordinate condition (1.2.2) to be satisfied at time $t = 0$.
- We solve (1.2.3) with such initial data, which is possible if they are regular enough. However we may a priori have existence only up to some time $T > 0$.
- We now have to check that the metric g, whose coefficients are given by the solution of (1.2.3) is indeed a solution to Einstein equations.

Let us describe this last point. We need to show that the wave coordinate condition holds true as long as the solution exists. We consider the metric g we have constructed: we can calculate its Ricci tensor

$$\begin{aligned} R_{\nu\nu} &= -\frac{1}{2}\Box_g g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + P_{\mu\nu}(g)(\partial g, \partial g), \\ &= \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right), \end{aligned}$$

where we have used the wave equation satisfied by g. Let us now recall the contracted Bianchi identity, which are always satisfied by the Einstein tensor of a metric

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0.$$

Replacing the Ricci tensor by its expression in term of H, we obtain for H a system of linear, homogeneous wave equations with variable coefficients. By uniqueness of the solution to such a system, it is therefore sufficient to check that the initial data vanish at time $t = 0$. The condition $H^\rho|_{t=0} = 0$ is ensured by the choice of $\partial_t g_{00}$ and $\partial_t g_{0I}$. To obtain the remaining condition, $\partial_t H^\rho|_{t=0} = 0$, we look at the constraint equations. We have that, at time $t = 0$

$$R_{00} - \frac{1}{2} R g_{00} = 0, \qquad R_{0i} = 0.$$

Again, the Ricci tensor can be expressed in term of ∂H, and here more precisely in term of $\partial_t H$, since H vanishes at time zero. The condition we obtain on $\partial_t H$ can be inverted, yielding the initial condition $\partial_t H^\rho|_{t=0} = 0$.

Wave coordinates have been used by Choquet-Bruhat to prove the local existence result [23].

1.2.3 Some fundamental results on the Cauchy problem for Einstein vaccum equations

As said before, Theorem 1.2.1 is a local existence result for regular data for Einstein vacuum equations. This opens the way to fundamental questions in partial differential equations like the issue of the minimal regularity needed to obtain local well-posedness, and the issue of global well-posedness.

In the first category, the most accurate result up to now is the bounded L^2 curvature theorem by Klainerman, Rodnianski and Szeftel [50], which says that Einstein equations are well posed at the level of the curvature in L^2.

In the second category, it is important to specify what we mean by global wellposedness. Indeed, singularities may form in general relativity, such as the singularity inside the Schwarschild black-hole. Another pathological behaviour which can happen is the formation of a Cauchy horizon, across which the solution could be continued in a non unique way, as in the Kerr black-hole. In these two particular examples the singularity is hidden inside a black-hole region: this is conjectured to be a generic behaviour in the so-called weak cosmic censorship, which can be seen as a global well-posedness conjecture for Einstein equations. This conjecture in its general form is still a far away goal. However, global well-posedness results have been proved in some perturbative cases. Let us quote a few results.

- The nonlinear stability of Minkowski has been proven by Christodoulou and Klainerman in [16]. Another proof, using wave coordinates, has been found by Lindblad and Rodnianski in [57].
- The nonlinear stability of Schwarzschild for axially symmetric perturbation has been proven by Klainerman and Szeftel in [51]. The full non linear stability of Schwarzschild (for data in a codimension 3 manifold) has been proven by Dafermos, Holzegel, Rodnianski and Taylor in [19].
- Some steps toward the nonlinear stability of Kerr for small angular momentum have been done by Klainerman and Szeftel in [52].

1.3 Overview of the results obtained by the author

1.3.1 List of results

Papers written after the PHD (in reverse chronological order):

[41] C. Huneau and C. Vâlcu : Einstein constraint equations for Kaluza-Klein spacetimes, *preprint*.

[40] C. Huneau and A. Stingo : Global well-posedness for a system of quasilinear wave equations on a product space, *preprint*.

[39] C. Huneau and J. Luk : Trilinear compensated compactness and Burnett's conjecture in general relativity, *preprint*.

[38] C. Huneau and J. Luk : High-frequency backreaction for the Einstein equations under polarized U(1)-symmetry, *Duke Mathematical Journal*.

[37] C. Huneau and J. Luk, Einstein equations under polarized U(1) symmetry in an elliptic gauge, *Communication in Mathematical Physics*.

[36] D. Häfner and C. Huneau, Instability of infinitely-many stationary solutions of the $SU(2)$ Yang-Mills fields on the exterior of the Schwarzschild black hole, *Advances in Differential Equations*.

[35] C. Huneau : Stability of Minkowski space-time with a translation space-like Killing field, *Annals of PDE*.

Papers written during the PHD:

[34] C. Huneau : Stability in exponential time of Minkowski space-time with a translation space-like Killing field, *Annals of PDE*.

[28] R. Gicquaud and C. Huneau : Limit equation for vacuum Einstein constraints with a translational Killing vector field in the compact hyperbolic case, *Journal of Geometry and Physics*.

[32] C. Huneau : Constraint equations for $3+1$ vacuum Einstein equations with a translational space-like Killing field in the asymptotically flat case. II, *Asymptotic Analysis*.

[31] C. Huneau : Constraint equations for $3+1$ vacuum Einstein equations with a translational space-like Killing field in the asymptotically flat case, *Annales Henri Poincaré*.

1.3.2 Brief description of the results

$\mathbb{U}(1)$ symmetry

Many results in the list above are in the context of $\mathbb{U}(1)$ symmetry (similar to a translation symmetry), so let us motivate briefly the introduction of this symmetry, presented more precisely in Chapter 2. This symmetry has the advantage of reducing Einstein vacuum equations in dimension $3+1$ to Einstein equations in dimension $2+1$ coupled to either a wave equation in the polarized case, or a wave-map system in the unpolarized case. This symmetry, by reducing the dimension, should also reduce the critical regularity needed to prove local well-posedness. A long standing conjecture says that Einstein equation in $\mathbb{U}(1)$ symmetry should be globally well-posed in the Sobolev space H^1. This conjecture has been proved, in presence of an additional symmetry in [3]. In [14], Choquet-Bruhat and Moncrief proved a global well-posedness result in H^2 in the case where space-time can be foliated by compact hyperbolic space-like surfaces in expansion.

Stability of Minkowski with a translation symmetry

A first natural global-existence question one can ask in the context of $\mathbb{U}(1)$ symmetry is the stability of the trivial solution, which can be formulated as the stability of Minkowski space-time of $3+1$ dimensions for perturbations that are symmetric in one of the space direction. More precisely, Einstein vacuum equations with polarized $\mathbb{U}(1)$ symmetry reduce to the following system coupling a Lorentzian metric g on $\mathbb{R}^{2+1}$ with a scalar field ϕ

$$\begin{cases} \Box_g \phi = 0 \\ R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi. \end{cases} \tag{1.3.1}$$

Let us note that if reducing the dimension is more favourable for regularity issues, it is known to be less favourable for stability, since the decay of the free wave is lower.

In $2+1$ dimensions, the constraint equation can be seen as a determined system. Consequently, the free initial data for (1.3.1) consist only of the initial data $(\phi, \partial_t \phi)\big|_{t=0} = (\phi_0, \phi_1)$ for the wave equation. The theorem proved in [35] is the following.

Theorem 1.3.1. *Let $0 < \varepsilon < 1$. Let $\frac{1}{2} < \delta < 1$ and $N \geq 25$. Let $(\phi_0, \phi_1) \in H^{N+2}(\mathbb{R}^2) \times H^{N+1}(\mathbb{R}^2)$ compactly supported in $B(0,R)$. We assume*

$$\|\phi_0\|_{H^{N+2}} + \|\phi_1\|_{H^{N+1}} \leq \varepsilon.$$

Let $\varepsilon \ll \rho \ll \sigma \ll \delta$, such that $\delta - 2\sigma > \frac{1}{2}$. If ε is small enough, there exists a global solution (g, ϕ) of (2.3.1). Moreover, if we call C the causal future of $B(0,R)$, and $\bar{C}$ its complement, there exists a coordinate system (t, x_1, x_2) in C and a coordinate system (t', x'_1, x'_2) in $\bar{C}$ such that we have in C :

$$(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1),$$
$$|g - m| \lesssim \frac{\varepsilon}{(1+t+|x|)^{\frac{1}{2}-\rho}}, \quad |\phi| \lesssim \frac{\varepsilon}{(1+t+|x|)^{\frac{1}{2}}(1+|q|)^{\frac{1}{2}-4\rho}},$$
$$\|\mathbb{1}_C \partial \phi\|_{H^{N+1}(\mathbb{R}^2)} + \|\mathbb{1}_C (1+t-|x|)^{-\frac{1}{2}-\sigma} \partial (g-m)\|_{H^N(\mathbb{R}^2)} \lesssim \varepsilon (1+t)^{2\rho},$$

where we note $q = r - t$, and we have in $\bar{C}$

$$\|\mathbb{1}_{\bar{C}} (1+|x|-t|)^{1+\delta-2\sigma} \partial (g - g_{\mathfrak{a}})\|_{H^N(\mathbb{R}^2)} \lesssim \varepsilon (1+t)^{2\rho},$$

where $g_{\mathfrak{a}}$ is defined by

$$g_{\mathfrak{a}} = -dt^2 + dr^2 + (r + \chi(q) a(\theta) q)^2 d\theta^2 + J(\theta)\chi(q) dq d\theta,$$

with χ a cut-off function equal to 1 for $q \geq 2$ and equal to 0 for $q \leq 1$ and

$$a(\theta) = a_0 + a_1 \cos(\theta) + a_2 \sin(\theta).$$

The parameter $a_0, (a_1, a_2), J$ are respectively the asymptotic deficit angle, linear momentum and angular momentum, and are determined by the initial data (see [32]).

In [35], we give also another version of this theorem, in a global coordinate chart : the price to pay is a more complicated formula for the asymptotic behaviour of the metric.

The main difficulty of the proof is due to the decay of the free wave equation in dimension $2+1$, which is only $\frac{1}{\sqrt{t}}$. With this decay, solutions to a system of wave equations with quadratic or cubic non-linearities may blow up if there is no structure. The main ingredient of the proof is to carefully choose a generalized wave gauge, that is to say a gauge where the coordinates satisfy an equation of the form $\Box_g x^\alpha = F^\alpha$, with F^α chosen to remove the quadratic terms which do not have the null structure. After this transformation, we are left with a system of wave equations with a weak cubic null structure, for which the global existence proof has some common features with the proof of the stability of Minkowski in wave coordinates by Lindblad and Rodnianski [57].

High-frequency limit in $\mathbb{U}(1)$ symmetry

The articles [37], [38] and [39], in collaboration with Jonathan Luk, are concerned with the behaviour of high-frequency solutions to Einstein-vacuum equation. To study such a behaviour, the $\mathbb{U}(1)$ symmetry, which is more favourable in term of the regularity needed, is particularly adapted. This research topic is the subject of Chapter 2.

Spherically symmetric $SU(2)$ Yang-Mills equation on the exterior of the Schwarzschild black hole

Studying Einstein equations with spherical symmetry could be an interesting toy model. However, Birkhoff's theorem (see Chapter 4 of [11]) tells us that the only solutions to Einstein vacuum equations with spherical symmetry are given by the Schwarzschild solutions. Therefore, to have a dynamical problem in spherical symmetry, one should consider Einstein equations coupled to another matter field. For instance in [15] Christodoulou considered Einstein scalar field equations with spherical symmetry, and subsequently proved the weak cosmic censorship in this setting.

Another interesting model is given by Einstein-Yang Mills equations with spherical symmetry: in that case even the decoupled equations are non linear. The asymptotic behaviour of solutions to such a system is not straightforward at all, due to the presence of a countable family of non trivial static solutions: see [63] for the existence of smooth solutions, and [64] for the black-holes solutions (see also [6] for a numerical study). These non trivial solutions are conjectured to be unstable [7].

In [36], a work in collaboration with Dietrich Häfner, we study the simplified uncoupled problem, that is to say a Yang Mills equation with spherical symmetry on a fixed Schwarzschild exterior. The equation we study takes the simple form

$$\partial_t^2 W - \partial_x^2 W + \frac{(1-\frac{2m}{r})}{r^2} W(W^2-1) = 0, \tag{1.3.2}$$

where the coordinates x and r are related by

$$\frac{dx}{dr} = \left(1 - \frac{2m}{r}\right)^{-1}.$$

The exterior of the black-hole is the region $r > 2m$. The range of x is $]-\infty, +\infty[$. There is a conserved energy for (1.3.2) given by

$$\mathcal{E}(W, \dot{W}) = \int \dot{W}^2 + (W')^2 + \frac{1-\frac{2m}{r}}{2r^2}(W^2-1)^2 dx.$$

The solutions of zero energy correspond to $W = \pm 1$. Perturbations of these trivial solutions decay to zero when the time goes to infinity (see [27]).

In [36], we study perturbations around the non trivial static solutions given by the following theorem

Theorem 1.3.2. *There exists a decreasing sequence* $\{a_n\}_{n\in\mathbb{N}^{\geq 1}}$, $0 < ... < a_n < a_{n-1} < ... < a_1 = \frac{1+\sqrt{3}}{3\sqrt{3}+5}$ *and, for all* $n \geq 1$*, smooth stationary solutions* W_n *of* (1.3.2) *with*

$$-1 \leq W_n \leq 1, \quad \lim_{x\to-\infty} W_n(x) = a_n, \quad \lim_{x\to\infty} W_n(x) = (-1)^n.$$

The solution W_n *has exactly* n *zeros.*

Remark 1.3.3. *There is an explicit formula for the first stationary solution (see [7])*

$$W_1(x) = \frac{c - \frac{r}{2m}}{\frac{r}{2m} + 3(c-1)}, \quad c = \frac{3+\sqrt{3}}{2}.$$

This solution corresponds to $\lim_{x\to-\infty} W_1(x) = a_1 = \frac{1+\sqrt{3}}{3\sqrt{3}+5}$.

We give a detailed proof of this result in the appendix of [36], where we follow arguments of Smoller, Wasserman, Yau and McLeod [63] and [64]. The main theorem of [36] states that the above solutions are all nonlinearly unstable in the energy space $\mathcal{E} = \mathcal{H}^1 \times L^2$ where

$$\|u\|^2_{\mathcal{H}^1} = \int (\partial_x u)^2 dx + \left(\int \frac{1-\frac{2m}{r}}{r^2} u^4 dx\right)^{\frac{1}{2}}.$$

Theorem 1.3.4. *For any* $n \geq 1$ *the solution* W_n *of* (1.3.2) *is unstable. More precisely, there exist* $\epsilon_0 > 0$ *and a sequence* $(W^m_{0,n}, W^m_{1,n})$ *with* $\|(W^m_{0,n}, W^m_{1,n}) - (W_n, 0)\|_{\mathcal{E}} \to 0$, $m \to \infty$*, but for all* m*, the solution* $W^m_n(t)$ *of* (1.3.2) *with initial data* $(W^m_{0,n}, W^m_{1,n})$ *satisfies*

$$\sup_{t\geq 0} \|(W^m_n(t), \partial_t W^m_n(t)) - (W_n, 0)\|_{\mathcal{E}} \geq \epsilon_0 > 0.$$

This instability is due to the presence of at least an unstable mode for the linearised equation around W_n, which consists of a wave equation with a potential V_n. The difficulty in proving analytically the presence of an unstable mode, is that the stationary solution W_n are only obtained with a shooting argument. Therefore we need to study more precisely the behaviour of the solutions W_n to obtain estimates on the potential V_n.

Einstein equations on space-time with additional compact directions

In some physical theories, such as supergravity (see [9]), Einstein equations are considered on manifold of the form $\mathcal{M} \times K$, where $\mathcal{M}$ is a $3+1$ manifold and K a compact manifold. The articles [41], in collaboration with Caterina Vâlcu, and [40], in collaboration with Annalaura Stingo, are preliminary studies for the stability of the trivial solution $(\mathbb{R}^{3+1} \times \mathbb{T}^d, \zeta)$ with $\zeta = m + (d\theta)^2$, where m is the Minkowski metric and $(d\theta)^2$ the flat metric on the torus $\mathbb{T}^d$. This research topic is the subject of Chapter 3.

Chapter 2

Burnett's conjecture and its reverse in $\mathbb{U}(1)$ symmetry

In general relativity, space-time is dynamical: the generic evolution of stellar objects produces gravitational radiation. When this gravitational radiation is weak, it can be modelled by studying the linearised Einstein equations around a particular solution. However, if the radiation is not weak, the non-linear features of Einstein equations begin to play a role and the problem becomes much more difficult. In 1968, Isaacson initiated a new perturbative scheme, inspired by WKB analysis, to study gravitational radiation in the limit of small amplitude but high frequency (see [43] and [44]). Around the same time, formal solutions have been computed by Yvonne Choquet Bruhat in the language of geometrical optics (see [10]).

In 1989, Burnett formulated a conjecture on the possible effects of small amplitude and high frequency perturbations in general relativity. For this, he considered a special regime where a sequence of metrics g_λ, solutions to Einstein vacuum equations is such that

- $g_\lambda \to g_0$ when $\lambda \to 0$, where g_0 is a smooth Lorentzian metric,
- ∂g_λ is bounded in L^∞_{loc}.

Because of the non-linearities in Einstein equation, the metric g_0 is not in general a solution to Einstein vacuum equations: terms of the form $(\partial g_\lambda)^2$ lead to a defect of convergence and we can write

$$(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})(g_0) = T^{eff}_{\mu\nu},$$

where we have put all the defect of convergence in the right-hand side, and called it the effective stress-energy tensor $T^{eff}_{\mu\nu}$. This effect is called the backreaction. In [8], Burnett conjectured that $T^{eff}_{\mu\nu}$ corresponds to the stress-energy tensor of a massless Vlasov field. This means that there should exist a non negative function $f(x,p)$ defined on the the null mass shell $P^*\mathcal{M} = \{(x,p) \in T^*\mathcal{M}, g_0^{-1}(p,p) = 0\}$ such that

$$T^{eff}_{\mu\nu} = \int_{P^*_x\mathcal{M}} f(x,p)p_\mu p_\nu d\sigma_{P^*_x\mathcal{M}},$$

where $d\sigma_{P^*_x\mathcal{M}}$ is the measure induced by g on $P^*_x\mathcal{M}$. Moreover, the function f should satisfy the transport equation

$$g_0^{\alpha\beta} p_\beta \partial_{x^\alpha} f - \frac{1}{2}\partial_{x^i} g_0^{\alpha\beta} p_\alpha p_\beta \partial_{p_i} f = 0.$$

In [8], Burnett also asked the reverse question : for a solution (g_0, f) to Einstein-massless Vlasov equations, can we found a one parameter family of metrics g_λ, solutions to Einstein vacuum equations such that g_λ converges uniformly to g_0, and the derivatives of g_λ are uniformly bounded?

To illustrate his conjecture, Burnett proposed the following example. Consider a sequence of plane wave solutions

$$g_\lambda = -dudv + B_\lambda(u)^2(e^{\omega_\lambda(u)}dx^2 + e^{-\omega_\lambda(u)}dy^2).$$

with the function ω_λ chosen to be of the form $\omega_\lambda(u) = \lambda\alpha(u)\cos\left(\frac{u}{\lambda}\right)$. Einstein vacuum equations are equivalent to the ordinary differential equation for B_λ

$$B_\lambda''(u) - \omega_\lambda(u)^2 B_\lambda(u) = 0.$$

Taking the high frequency limit, that is to say letting $\lambda \to 0$ we obtain $g_\lambda \to g_0$ uniformly, where

$$g_0 = -dudv + B_0(u)^2(dx^2 + dy^2),$$

and

$$B_0''(u) - \frac{1}{2}\alpha(u)^2 B_0(u) = 0.$$

We can compute that g_0 is no longer a solution to Einstein vacuum equations. Instead we have

$$R_{\mu\nu}[g_0] = \frac{1}{2}\alpha(u)^2 u_\mu u_\nu.$$

The right hand-side corresponds to the stress-energy tensor of a null dust, that is to say a massless fluid without pressure. This can be seen as a particular case of a Vlasov field.

This problem has been subsequently revisited in the context of perturbations in cosmology (see [29]). In cosmology, the universe is often modelled by a Friedman-Lemaitre-Roberston-Walker space-time, which is a homogeneous and isotropic solution to Einstein-Euler equations. However this model have some fitting issue with observations, which are commonly solved by the introduction of dark matter and dark energy. Moreover, the homogeneity is a large scale property, which is not true at small scale, and taking averages does not commute with taking products, as explained in [21] : this led to the idea that inhomogeneities at small scale could affect the behaviour of solution at large scale, and might mimic the effect of dark energy.

In [29], Green and Wald answered this question in the following setting. They consider a one parameter family of metrics g_λ, solutions to Einstein equations with a stress-energy tensor $T_{\mu\nu}$ satisfying the weak energy condition, which is that for all time-like vector field X, we have $T_{\mu\nu}X^\mu X^\nu \geq 0$. Assume also that there exists g_0 such that $\frac{1}{\lambda}|g_\lambda - g_0|$ and $|\partial g_\lambda|$ are uniformly bounded in L^∞_{loc}, and that products of derivatives of g_λ have a weak limit. Then the effective stress-energy tensor is traceless and satisfies the weak energy condition, which excludes the creation of dark energy. The right setting for the study of small scale inhomogeneities in cosmology is however still subject to debates.

In a series of joint works with Jonathan Luk, which will be presented in this chapter, we study Burnett's conjecture and its reverse in $U(1)$ symmetry. The advantage of this setting is the possibility to work in an elliptic gauge, which allows to say that some components have a better behaviour in term of oscillations.

Burnett's conjecture and its reverse have also been studied by Luk and Rodnianski in [58]. They work in $3+1$ dimension, without symmetry assumptions, but on solutions with higher regularity in angular directions, expressed in a double null foliation.

2.1 Einstein equations with $\mathbb{U}(1)$ symmetry and elliptic gauge

In the whole chapter, we will work in a $(3+1)$-dimensional manifold ${}^{(4)}\mathcal{M} = \mathcal{M} \times \mathbb{R}$, where $\mathcal{M} = (0,T) \times \mathbb{R}^2$. Introduce coordinates (t, x^1, x^2) on $\mathcal{M}$ and (t, x^1, x^2, x^3) on ${}^{(4)}\mathcal{M}$ in the obvious manner.

Consider a Lorentzian metric ${}^{(4)}g$ on ${}^{(4)}\mathcal{M}$ with a $\mathbb{U}(1)$ symmetry, i.e. ${}^{(4)}g$ takes the form

$$ {}^{(4)}g = e^{-2\psi} g + e^{2\psi}(\mathrm{d}x^3 + \mathfrak{A}_\alpha \mathrm{d}x^\alpha)^2, \tag{2.1.1} $$

where g is a Lorentzian metric on $\mathcal{M}$, ψ is a real-valued function on $\mathcal{M}$ and $\mathfrak{A}_\alpha$ is a real-valued 1-form on $\mathcal{M}$.

Under these assumptions, it is well known that the Einstein vacuum equations for $({}^{(4)}\mathcal{M}, {}^{(4)}g)$ reduces to the following $(2+1)$-dimensional Einstein–wave map system for $(\mathcal{M}, g, \psi, \varpi)$ (see for instance [11]),

$$ \begin{cases} \Box_g \psi + \frac{1}{2} e^{-4\psi} g^{-1}(\mathrm{d}\varpi, \mathrm{d}\varpi) = 0, \\ \Box_g \varpi - 4 g^{-1}(\mathrm{d}\varpi, \mathrm{d}\psi) = 0, \\ \mathrm{R}_{\alpha\beta}(g) = 2\partial_\alpha\psi\partial_\beta\psi + \frac{1}{2} e^{-4\psi} \partial_\alpha\varpi\partial_\beta\varpi, \end{cases} \tag{2.1.2} $$

where ϖ is a real-valued function which relates to $\mathfrak{A}_\alpha$ via the relation

$$ (\mathrm{d}\mathfrak{A})_{\alpha\beta} = \partial_\alpha \mathfrak{A}_\beta - \partial_\beta \mathfrak{A}_\alpha = \frac{1}{2} e^{-4\psi} (g^{-1})^{\lambda\delta} \in_{\alpha\beta\lambda} \partial_\delta \varpi, \tag{2.1.3} $$

where $\in_{\alpha\beta\lambda}$ denotes the completely antisymmetric tensor. The function ϖ is called the twist potential.

2.1.1 Elliptic gauge

We write the $(2+1)$-dimensional metric g on $\mathcal{M} := I \times \mathbb{R}^2$ in the form

$$ g = -N^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{2.1.4} $$

Let $\Sigma_t := \{(s, x^1, x^2) : s = t\}$ and $e_0 = \partial_t - \beta^i \partial_i$, which is a future directed normal to Σ_t. We introduce the second fundamental form of the embedding $\Sigma_t \subset \mathcal{M}$

$$ K_{ij} = -\frac{1}{2N} \mathcal{L}_{e_0} \bar{g}_{ij}. \tag{2.1.5} $$

We decompose K into its trace and traceless parts.

$$ K_{ij} =: H_{ij} + \frac{1}{2} \bar{g}_{ij} \tau. \tag{2.1.6} $$

Here, $\tau := \mathrm{tr}_{\bar{g}} K$ and H_{ij} is therefore traceless with respect to $\bar{g}$.

Introduce the following *gauge conditions*:

- $\bar{g}$ is conformally flat, i.e., for some function γ : $\bar{g}_{ij} = e^{2\gamma}\delta_{ij}$;
- The constant t-hypersurfaces Σ_t are maximal : $\tau = 0$.

Under this gauge conditions, the metric components N, γ and β^i satisfy the following elliptic equations; see [37, Appendix B]:

$$\delta^{ik}\partial_k H_{ij} = -\frac{e^{2\gamma}}{N} R_{0j}, \tag{2.1.7}$$

$$\Delta\gamma = -\frac{e^{2\gamma}}{N^2} G_{00} - \frac{1}{2} e^{-2\gamma}|H|^2, \tag{2.1.8}$$

$$\Delta N = N e^{-2\gamma}|H|^2 - \frac{1}{2} e^{2\gamma} N R + \frac{e^{2\gamma}}{N} G_{00}, \tag{2.1.9}$$

$$(\mathfrak{L}\beta)_{ij} = 2N e^{-2\gamma} H_{ij}, \tag{2.1.10}$$

where $R_{\alpha\beta}$ is the Ricci tensor, R is the scalar curvature, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ is the Einstein tensor. In the case of (2.1.2), these terms can be computed using

$$\mathrm{R}_{\alpha\beta}(g) = 2\partial_\alpha\psi\partial_\beta\psi + \frac{1}{2} e^{-4\psi}\partial_\alpha\varpi\partial_\beta\varpi.$$

The conformal Killing operator $\mathfrak{L}$ is given by

$$(\mathfrak{L}\beta)_{ij} := \delta_{j\ell}\partial_i\beta^\ell + \delta_{i\ell}\partial_j\beta^\ell - \delta_{ij}\partial_k\beta^k. \tag{2.1.11}$$

Taking the divergence of (2.1.10) and using (2.1.7) we can write an elliptic equation for β whose principal symbol is Δ.

Moreover, the spatial components of the Ricci tensor is given by (see [37, Proposition B2])

$$\begin{aligned} R_{ij} = & \delta_{ij}\left(-\Delta\gamma - \frac{1}{2N}\Delta N\right) - \frac{1}{N}(\partial_t - \beta^k\partial_k) H_{ij} - 2e^{-2\gamma} H_i{}^\ell H_{j\ell} \\ & + \frac{1}{N}\left(\partial_j\beta^k H_{ki} + \partial_i\beta^k H_{kj}\right) - \frac{1}{N}\left(\partial_i\partial_j N - \frac{1}{2}\delta_{ij}\Delta N - \left(\delta_i^k\partial_j\gamma + \delta_j^k\partial_i\gamma - \delta_{ij}\delta^{\ell k}\partial_\ell\gamma\right)\partial_k N\right). \end{aligned} \tag{2.1.12}$$

2.1.2 Local well-posedness in elliptic gauge for Einstein null dust system

The way to prove that a gauge choice is admissible, is to show local well-posedness with this gauge condition. It is the strategy of [37], where we Einstein null dust system in polarized $\mathbb{U}(1)$ symmetry. A null dust is a fluid which is isotropic (for instance a fluid of photons), and without pressure. The polarized character means that we take $\mathfrak{A}_\alpha = 0$, and replace the wave-map system by a single wave equation. We can write Einstein null dust system under the form

$$\begin{cases} R_{\mu\nu}(g) = 2\partial_\mu\psi\partial_\nu\psi + \sum_{\mathbf{A}}(F_{\mathbf{A}})^2\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{A}}, \\ \Box_g\psi = 0, \\ 2(g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta F_{\mathbf{A}} + (\Box_g u_{\mathbf{A}})F_{\mathbf{A}} = 0, \\ (g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta u_{\mathbf{A}} = 0. \end{cases} \tag{2.1.13}$$

where the index $\mathbf{A}$ belong to some finite set. The purpose of this study is to be applied to our result in [38], described in Section 2.2, which says that we can approach any small and regular solution to Einstein null dust system in $\mathbb{U}(1)$ symmetry by a one parameter family of solutions to Einstein vacuum equations in $\mathbb{U}(1)$ symmetry $h_\lambda = (g_\lambda, \psi_\lambda)$ such that ∂h_λ is bounded in L^∞ and $\|\partial^2 h_\lambda\|_{L^2} \leq \lambda^{-1}$. Consequently, our local well posedness result will be two folds.

- We want to show local well posedness for Einstein null dust system, for small and regular initial data.
- For Einstein vacuum equations, the local well-posedness result should not impose smallness for more than one derivative of ψ.

The initial data

Since we will be dealing with elliptic equations, it is natural to introduce weighted Sobolev spaces, which are particularly well adapted to the resolution of the Laplace equation.

Definition 2.1.1. *Let $m \in \mathbb{N}$, $1 < p < \infty$, $\delta \in \mathbb{R}$. The weighted Sobolev space $W^m_{\delta,p}$ is the completion of C^∞_0 under the norm*

$$\|u\|_{W^m_{\delta,p}} = \sum_{|\beta| \leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} \nabla^\beta u\|_{L^p}.$$

We will use the notation $H^m_\delta = W^m_{\delta,2}$.

The initial data for (2.1.13) consist of the prescription of the geometry (first and second fundamental forms of Σ_0) as well as the matter fields. For convenience, we will require $\nabla\psi$, the normal derivative of ψ and $F_{\mathbf{A}}$ to be initially compactly supported. By the finite speed of propagation, they will remain spatially compactly supported.

To completely specify the initial data, we also need to prescribe the initial values for solutions to the eikonal equation $(g^{-1})^{\mu\nu}\partial_\mu u_{\mathbf{A}} \partial_\nu u_{\mathbf{A}} = 0$. To this end, we will prescribe the initial values for $u_{\mathbf{A}}\big|_{\Sigma_0}$ and will require also that

1. $\min_{\mathbf{A}} \inf_{x\in\mathbb{R}^2} \left|\nabla u_{\mathbf{A}}\big|_{\Sigma_0}\right|(x) > C_{eik}^{-1}$ for some $C_{eik} > 0$,
2. $(e_0 u_{\mathbf{A}})\big|_{\Sigma_0} > 0$, $\forall \mathbf{A}$.

Moreover, while $u_{\mathbf{A}}$ is only physically relevant on the support of $F_{\mathbf{A}}$ (see (2.1.13)), we will for technical convenience define $u_{\mathbf{A}}$ globally on the initial hypersurface Σ_0 and also require the level sets of $u_{\mathbf{A}}$ to be asymptotic to planes in $\mathbb{R}^2$, or more precisely, for each $\mathbf{A} \in \mathcal{A}$, there exists a constant vector field $\overrightarrow{c_{\mathbf{A}}}$ such that $\nabla u_{\mathbf{A}} - \overrightarrow{c_{\mathbf{A}}}$ is in an appropriate weighted Sobolev space.

Before we proceed to define the notion of admissible initial data, we need to fix a cutoff function for the rest of the section:

Definition 2.1.2 (Cutoff function χ)**.** *From now on, let $\chi(|x|)$ be a fixed smooth cutoff function with $\chi = 0$ for $|x| \leq 1$ and $\chi = 1$ for $|x| \geq 2$.*

We now make precise the discussions on the initial data set in the following definition:

Definition 2.1.3 (Admissible initial data)**.** *For $-\frac{1}{2} < \delta < 0$, $k \geq 3$, $R > 0$ and $\mathcal{A}$ a finite set, an* **admissible initial data set** *with respect to the elliptic gauge for* (2.1.13) *consists of*

1. *a conformally flat intrinsic metric* $e^{2\gamma}\delta_{ij}\big|_{\Sigma_0}$ *which admits a decomposition*
$$\gamma = -\alpha\chi(|x|)\log(|x|) + \widetilde{\gamma},$$
where $\alpha \geq 0$ *is a constant,* $\chi(|x|)$ *is as in Definition 2.1.2, and* $\tilde{\gamma} \in H^{k+2}_\delta$*;*
2. *a second fundamental form* $(H_{ij})\big|_{\Sigma_0} \in H^{k+1}_{\delta+1}$ *which is traceless;*
3. $(\frac{1}{N}(e_0\psi), \nabla\psi)\big|_{\Sigma_0} \in H^k$*, compactly supported in* $B(0,R)$*;*
4. $F_{\mathbf{A}}\big|_{\Sigma_0} \in H^k$*, compactly supported in* $B(0,R)$ *for every* $\mathbf{A} \in \mathcal{A}$*;*
5. $u_{\mathbf{A}}\big|_{\Sigma_0}$ *such that* $\inf_{x\in\mathbb{R}^2}\left|\nabla u_{\mathbf{A}}\big|_{\Sigma_0}\right|(x) > C_{eik}^{-1}$ *for some* $C_{eik} > 0$ *and* $\left(\nabla u_{\mathbf{A}}\big|_{\Sigma_0} - \overrightarrow{c_{\mathbf{A}}}\right) \in H^{k+1}_\delta$*, where* $\overrightarrow{c_{\mathbf{A}}}$ *is a constant vector field for every* $\mathbf{A} \in \mathcal{A}$*.*

Moreover γ *and* H *are required to satisfy the following* **constraint equations***:*
$$\delta^{ik}\partial_k H_{ij} = -\frac{2e^{2\gamma}}{N}(e_0\psi)\partial_j\psi - \sum_{\mathbf{A}} e^{\gamma}F_{\mathbf{A}}^2|\nabla u_{\mathbf{A}}|\partial_j u_{\mathbf{A}}, \tag{2.1.14}$$
$$\Delta\gamma + e^{-2\gamma}\left(\frac{e^{4\gamma}}{N^2}(e_0\psi)^2 + \frac{1}{2}|H|^2\right) + |\nabla\psi|^2 + \sum_{\mathbf{A}} F_{\mathbf{A}}^2|\nabla u_{\mathbf{A}}|^2 = 0. \tag{2.1.15}$$

Let us note that the term in $-\alpha\chi(|x|)\log(|x|)$ in the expression of γ comes from the fundamental solution to the Laplace operator in $\mathbb{R}^2$, and the fact that γ satisfies (2.1.15).

It turns out that we can find freely prescribable initial data, from which (under suitable smallness assumptions) one can construct admissible initial data satisfying the constraint equations. To this end, it will be convenient not to prescribe the unit normal derivative $\frac{1}{N}e_0$ of the scalar field ψ and the density of the null dusts $F_{\mathbf{A}}$, but instead prescribe appropriately rescaled versions. More precisely, define $\dot{\psi}$, $\breve{F}_{\mathbf{A}}$ by
$$\dot{\psi} = \frac{e^{2\gamma}}{N}(e_0\psi), \quad \breve{F}_{\mathbf{A}} = F_{\mathbf{A}}e^{\frac{\gamma}{2}}. \tag{2.1.16}$$

We define the notion of admissible free initial data as follows:

Definition 2.1.4 (Admissible free initial data)**.** *For* $-\frac{1}{2} < \delta < 0$*,* $k \geq 3$*,* $R > 0$ *and* $\mathcal{A}$ *a finite set, an* **admissible free initial data set** *with respect to the elliptic gauge is given by the following:*

1. $(\dot{\psi}, \nabla\psi)\big|_{\Sigma_0} \in H^k$*, compactly supported in* $B(0,R)$*;*
2. $\breve{F}_{\mathbf{A}}\big|_{\Sigma_0} \in H^k$*, compactly supported in* $B(0,R)$ *for every* $\mathbf{A} \in \mathcal{A}$*;*
3. $u_{\mathbf{A}}\big|_{\Sigma_0}$ *such that* $\inf_{x\in\mathbb{R}^2}\left|\nabla u_{\mathbf{A}}\big|_{\Sigma_0}\right|(x) > C_{eik}^{-1}$ *for some* $C_{eik} > 0$ *and* $\left(\nabla u_{\mathbf{A}}\big|_{\Sigma_0} - \overrightarrow{c_{\mathbf{A}}}\right) \in H^{k+1}_\delta$*, where* $\overrightarrow{c_{\mathbf{A}}}$ *is a constant vector field for every* $\mathbf{A} \in \mathcal{A}$*.*

Moreover, $(\dot{\psi}, \nabla\psi, \breve{F}_{\mathbf{A}}, u_{\mathbf{A}})\big|_{\Sigma_0}$ *is required to satisfy*
$$\int_{\mathbb{R}^2}\left(-2\dot{\psi}\partial_j\psi - \sum_{\mathbf{A}}\breve{F}_{\mathbf{A}}^2|\nabla u_{\mathbf{A}}|\partial_j u_{\mathbf{A}}\right)dx = 0. \tag{2.1.17}$$

In order to obtain an admissible initial data set from an admissible free initial data set, one then solves the constraint equations (2.1.14) and (2.1.15), and also solves the equations (2.1.8) and (2.1.9) for N and β. We remark that (2.1.17) is the integrability condition necessary to solve (2.1.14).

Under suitable smallness assumptions, an admissible free initial data set gives rise to an actual admissible initial data set satisfying the constraint equations (see [37]).

Local well posedness result

The following theorem is the main result of [37]. Its content is the local well posedness in elliptic gauge for (2.1.13).

Theorem 2.1.5. *Let $-\frac{1}{2} < \delta < 0$, $k \geq 3$, $R > 0$ and $\mathcal{A}$ be a finite set. Given an admissible free initial data set as in Definition 2.1.4 such that*

$$\|\dot{\psi}\|_{L^\infty} + \|\nabla\psi\|_{L^\infty} + \max_{\mathbf{A}} \|\breve{F}_{\mathbf{A}}\|_{L^\infty} \leq \varepsilon, \tag{2.1.18}$$

$$C_{eik} := \left(\min_{\mathbf{A}} \inf_{x\in\mathbb{R}^2} |\nabla u_{\mathbf{A}}|(x)\right)^{-1} + \max_{\mathbf{A}} \|\nabla u_{\mathbf{A}} - \overrightarrow{c_{\mathbf{A}}}\|_{H^{k+1}_\delta} < \infty, \tag{2.1.19}$$

and

$$C_{high} := \|\dot{\psi}\|_{H^k} + \|\nabla\psi\|_{H^k} + \|\breve{F}_{\mathbf{A}}\|_{H^k} < \infty. \tag{2.1.20}$$

Then, for any C_{eik} and C_{high}, there exists a constant $\varepsilon_{low} = \varepsilon_{low}(C_{eik}, k, \delta, R) > 0$ independent of C_{high} and a $T = T(C_{high}, C_{eik}, k, \delta, R) > 0$ such that if $\varepsilon < \varepsilon_{low}$, there exists a unique solution to (2.1.13) in elliptic gauge on $[0,T] \times \mathbb{R}^2$. Moreover, the metric components γ and N can be decomposed as

$$\gamma = \alpha\chi(|x|)\log(|x|) + \widetilde{\gamma}, \quad N = 1 + N_{asymp}(t)\chi(|x|)\log(|x|) + \widetilde{N},$$

where $\alpha \leq 0$ is a constant, $N_{asymp}(t) \geq 0$ is a function of t alone, and $\chi(|x|)$ is as in Definition 2.1.2, $\tilde{\gamma}, \tilde{N} \in H^{k+2}_\delta$.

The smallness requirement is not so usual for a local well posedness result. In fact it is already required in order to solve the constraint equations (see also [33]). It is fundamental for our applications that the smallness requirement does not depend on the higher order norms (that is on the constant C_{high}).

An important fact to notice about the solution, is that the quantity α, which appears in the decomposition of γ is conserved. This quantity has a geometric meaning : it is the defect angle at infinity of the metric. It can be seen as the two dimensional equivalent of the ADM mass. The conservation of the defect angle can be interpreted as the conservation of the total energy of the system. As a consequence of our result, we have an other conservation law which is that the equality (2.1.17) holds for all times. This can be interpreted as the conservation of total linear momentum.

Strategy of the proof

The main difficulties in proving Theorem 2.1.5 comes from the fact that the conservation of the total energy and linear momentum are necessary to solve (2.1.13) in elliptic gauge, but they do not seem to be provable directly from the equations of motion. However, since Einstein equation are overdetermined, we have additional relations between the metric components which allows to change the nature of some of the equations.

- Instead of solving the elliptic equation for γ, given by (2.1.8) we can solve a wave equation : then the asymptotic behaviour in $\alpha \ln(|x|)$ is automatically conserved.
- Instead of solving the elliptic equation for H, given by (2.1.7) we can solve a transport equation, obtained from (2.1.12). Then the equality (2.1.17) is not required for the resolution.

The presence of the null dusts brings a specific difficulty which is an apparent loss of derivative. The classical way of solving the eikonal equation is to replace it by the geodesic equation for $L_\alpha = \partial_\alpha u$. This implies that in terms of derivatives we have L at the level of one derivative of the metric g. From the transport equation for F, and the presence of the term $\Box_g u = div_g L$, we then have that F is at the level of a derivative of L, so two derivatives of the metric g. This is a problem since L appears in the right-hand-side of the equations for g. This problem can be solved by using in addition Raychaudhuri equation for $\chi_{\mathbf{A}} = \Box_g u_{\mathbf{A}}$.

We call the system we obtain at the end of this process the reduced system. The Einstein part is a mix of elliptic, wave and transport equations in the t direction. The equations of motions are still wave and transports equation in the $u_{\mathbf{A}}$ direction. The strategy to prove Theorem 2.1.5 is then the following.

- To show local well-posedness for the reduced system. This is done via an iterative scheme. The difficulty comes from the interplay between evolution equations, for which the convergence of the sequence of solutions can be obtained by taking a time which is small enough, and nonlinear elliptic equations, for which we need to use the smallness of some of the parameters to show the convergence. The fact that we track the dependency in ε and C_{high} leads us to use an involved hierarchy of estimates.
- To show that the solution obtained at the end is indeed a solution to Einstein equations in elliptic gauge. This is done by using Bianchi identities, the same which allow to conclude the local well-posedness in wave coordinates presented in Section 1.2.2.

2.1.3 An improved local well posedness result for the vacuum equations

In view of the application described in Section 2.3, we need a local well posedness result for Einstein equations in unpolarized $\mathbb{U}(1)$ symmetry, with a relaxed smallness criterion : the required assumption should be that the derivatives of ψ and ϖ are small in L^4 instead of L^∞. This result has been proved by Arthur Touati in [66]. His result is the following.

Theorem 2.1.6. *Let $k = 2$ and consider free intiaial data $(\dot{\psi}, \nabla\psi)$, $(\dot{\varpi}, \nabla\varpi)$ such that*

$$\|\dot{\psi}\|_{L^4} + \|\nabla\psi\|_{L^4} + \|\dot{\varpi}\|_{L^4} + \|\nabla\varpi\|_{L^4} \leq \varepsilon.$$

For ε small enough, there exists a time T and a unique solution to (2.1.2) *in elliptic gauge on $[0, T[\times\mathbb{R}^2$. Moreover, if the maximal time of existence T is finite, then either the H^1 norm of $(\dot{\psi}, \nabla\psi)$, $(\dot{\varpi}, \nabla\varpi)$ diverges, either the smallness in L^4 no longer holds.*

2.2 Construction of high-frequency space-times : the case of null dusts

The aim of [38] is to show that any small, regular and local in time solution to (2.1.13) satisfying the angular separation condition of Definition 2.2.1 can be approached by a one parameter family of solutions to Einstein vacuum equations in polarized $\mathbb{U}(1)$ symmetry

$$\begin{cases} \Box_g \psi = 0, \\ \mathrm{Ric}_{\alpha\beta}(g) = 2\partial_\alpha\psi\partial_\beta\psi, \end{cases} \tag{2.2.1}$$

Definition 2.2.1. *Given a solution to* (2.1.13) *on* $I \times \mathbb{R}^2$ *for* $I \subset \mathbb{R}$ *(which is as regular as that in Theorem 2.1.5), we say that the set of eikonal functions* $\{u_{\mathbf{A}}\}_{\mathbf{A}\in\mathcal{A}}$ *is* **angularly separated** *if there exists* $\eta' \in (0,1)$ *such that*

$$\frac{\delta^{ij}(\partial_i u_{\mathbf{A}_1})(\partial_j u_{\mathbf{A}_2})}{|\nabla u_{\mathbf{A}_1}||\nabla u_{\mathbf{A}_2}|}(t,x) < 1 - \eta', \quad \forall (t,x) \in I \times \mathbb{R}^2, \quad \forall \mathbf{A}_1 \neq \mathbf{A}_2.$$

The precise statement of our theorem is the following.

Theorem 2.2.2 (Main theorem)**.** *Let* $k \geq 10$, $-\frac{1}{2} < \delta < \delta + 3\varepsilon < 0$, $R > 0$ *and* $\mathcal{A}$ *be a finite set. Given an admissible free initial data set* $(\dot\psi, \nabla\psi, \breve{F}_{\mathbf{A}}, u_{\mathbf{A}})\big|_{\Sigma_0}$ *(see Definition 2.1.4), such that*

- $$|\nabla u_{\mathbf{A}}\big|_{\Sigma_0}| > \frac{1}{2};$$
- $u_{\mathbf{A}}\big|_{\Sigma_0}$ *is angularly separated;*
- *(Smallness condition)*
$$\|\nabla u_{\mathbf{A}} - \overrightarrow{c_{\mathbf{A}}}\|_{H^{k+1}_{\delta+3\varepsilon}} + \|\dot\psi\|_{H^{10}} + \|\nabla\psi\|_{H^k} + \|\breve{F}_{\mathbf{A}}\|_{H^k} \leq \varepsilon; \tag{2.2.2}$$
- *(Genericity condition on initial data) there exists a point* $p \in \mathbb{R}^2$ *such that*
$$\begin{pmatrix} (\partial_1\psi_0)\big|_{\Sigma_0} \\ (\partial_1\dot\psi_0)\big|_{\Sigma_0} \end{pmatrix}(p) \text{ and } \begin{pmatrix} (\partial_2\psi_0)\big|_{\Sigma_0} \\ (\partial_2\dot\psi_0)\big|_{\Sigma_0} \end{pmatrix}(p) \text{ are linearly independent.} \tag{2.2.3}$$

Then, there exists $\varepsilon_0 > 0$ *(depending on* k, δ, R, $|\mathcal{A}|$ *and* $\{\overrightarrow{c_{\mathbf{A}}}\}_{\mathbf{A}\in\mathcal{A}}$*) such that if* $\varepsilon < \varepsilon_0$*, a unique solution* $(g_0, \psi_0, (F_0)_{\mathbf{A}}, (u_0)_{\mathbf{A}})$ *to* (2.1.13) *arising from the given admissible free initial data set exists on a time interval* $[0,1]$*, and there exists a one-parameter family of solutions* $(g_\lambda, \psi_\lambda)$ *to* (2.3.1) *for* $\lambda \in (0, \lambda_0)$ *(for some* $\lambda_0 \in \mathbb{R}$ *sufficiently small), which are all defined on the time interval* $[0,1]$*, such that*

$$(g_\lambda, \psi_\lambda) \to (g_0, \psi_0) \quad \text{uniformly on compact sets}$$

and

$$(\partial g_\lambda, \partial\psi_\lambda) \rightharpoonup (\partial g_0, \partial\psi_0) \quad \text{weakly in } L^2$$

with $\partial g_\lambda, \partial\psi_\lambda \in L^\infty_{loc}$ *uniformly.*

Let us make a few remarks on this theorem.

- The existence of the background solution $(g_0, \psi_0, (F_0)_{\mathbf{A}}, (u_0)_{\mathbf{A}})$ is a direct consequence of Theorem 2.1.5.
- The angular separation condition appears at several places in our proof (see section 2.2.2)

- The genericity condition is a technical condition to ensure that we can take initial data for $(g_\lambda, \psi_\lambda)$ satisfying (2.1.17).

The proof of Theorem 2.2.2 may be more interesting than the result in itself, because in the process we construct a high frequency ansatz for a solution to (2.2.1) and show the superposition of high frequency waves, in the spirit of geometrical optics. The main ideas of the proof are explained in the next subsections.

2.2.1 Strategy of the proof

The strategy of the proof of Theorem 2.2.2 is to show the existence of solutions of (2.2.1) in elliptic gauge of the general form

$$\psi_\lambda = \psi_0 + \sum_{\mathbf{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \widetilde{\psi}_\lambda, \quad g_\lambda = g_0 + \widetilde{g}_\lambda, \tag{2.2.4}$$

where $\widetilde{\psi}_\lambda$ and $\widetilde{\mathfrak{g}}_\lambda$ are terms which are higher order in λ. To do so, the idea is

- to use the local well posedness result applied to oscillatory data to obtain the existence of a solution of the form (2.2.4), on a time interval shrinking when $\lambda \to 0$,
- to write a bootstrap argument on the remainder $(\widetilde{\psi}_\lambda, \widetilde{g}_\lambda)$ to show the existence of the solution with this ansatz up to time 1.

If we manage to prove the existence of solutions of the form (2.2.4), Theorem 2.2.2 follows since we have directly that $(g_\lambda, \psi_\lambda) \to (g_0, \psi_0)$, and moreover

$$\begin{aligned}
&\partial_\alpha \psi_\lambda \partial_\beta \psi_\lambda \\
=&\partial_\alpha \psi_0 \partial_\beta \psi_0 + \sum_{\mathbf{A},\mathbf{B}} F_{\mathbf{A}} F_{\mathbf{B}} \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{B}} + \partial_{(\alpha}\psi_0 \sum_{\mathbf{A}} F_{\mathbf{A}} \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \partial_{\beta)} u_{\mathbf{A}} + O(\lambda) \\
=&\partial_\alpha \psi_0 \partial_\beta \psi_0 + \frac{1}{2}\sum_{\mathbf{A}} F_{\mathbf{A}}^2 \left(1 + \cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right)\right)) \partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{A}} + \sum_{\mathbf{A}\neq\mathbf{B}} F_{\mathbf{A}} F_{\mathbf{B}} \cos\left(\frac{u_{\mathbf{A}} \pm u_{\mathbf{B}}}{\lambda}\right) \partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{B}} \\
&\quad + \partial_{(\alpha}\psi_0 \sum_{\mathbf{A}} F_{\mathbf{A}} \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \partial_{\beta)} u_{\mathbf{A}} + O(\lambda) \\
\rightharpoonup&\partial_\alpha \psi_0 \partial_\beta \psi_0 + \frac{1}{2}\sum_{\mathbf{A}} F_{\mathbf{A}}^2 \partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{A}},
\end{aligned}$$

so passing to the weak limit in (2.2.1) we obtain that (g_0, ψ_0) satisfies (2.1.13).

2.2.2 Parametrix construction

Parametrix for the metric coefficients

It turns out that the ansatz (2.2.4) is not precise enough to run our argument. In fact, this can be seen from the computation of $\partial_\alpha \psi_\lambda \partial_\beta \psi_\lambda$ above. Let us write schematically the elliptic equations satisfied by the metric coefficient under the form

$$\Delta g = (\partial \psi)^2 + |\nabla g|^2.$$

The equation for $g_\lambda - g_0$ is then

$$\Delta(g_\lambda - g_0) = \frac{1}{2}\sum_{\mathbf{A}} F_{\mathbf{A}}^2 \cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right)\partial u_{\mathbf{A}}\partial u_{\mathbf{A}} + \sum_{\mathbf{A}\neq\mathbf{B}} F_{\mathbf{A}}F_{\mathbf{B}}\cos\left(\frac{u_{\mathbf{A}}\pm u_{\mathbf{B}}}{\lambda}\right)\partial u_{\mathbf{A}}\partial u_{\mathbf{B}} + \partial\psi_0 \sum_{\mathbf{A}} F_{\mathbf{A}}\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right)\partial u_{\mathbf{A}} + O(\lambda)$$

In this equation, the $O(1)$ terms have explicit expressions, and can be cancelled by introducing an approximate solution of the form

$$\begin{aligned} g_1 = &-\frac{1}{8}\sum_{\mathbf{A}} \frac{\lambda^2 F_{\mathbf{A}}^2}{|\nabla u_{\mathbf{A}}|^2}(\partial u_{\mathbf{A}})(\partial u_{\mathbf{A}})\cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right) \\ &-\sum_{\mathbf{A}} \frac{\lambda^2 F_{\mathbf{A}}}{|\nabla u_{\mathbf{A}}|^2}(\partial\psi_0)(\partial u_{\mathbf{A}})\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \\ &-\sum_{\mathbf{B}\neq\mathbf{A}} \frac{(\mp 1)\cdot\lambda^2 F_{\mathbf{A}}F_{\mathbf{B}}}{|\nabla(u_{\mathbf{A}}\pm u_{\mathbf{B}})|^2}(\partial_\mu u_{\mathbf{A}})(\partial_\nu u_{\mathbf{B}})\cos\left(\frac{u_{\mathbf{A}}\pm u_{\mathbf{B}}}{\lambda}\right). \end{aligned} \tag{2.2.5}$$

In order to do so, we need to have $|\nabla u_{\mathbf{A}}|^2$ and $|\nabla(u_{\mathbf{A}} \pm u_{\mathbf{B}})|^2$ to be greater than some positive constant. The first condition is assumed in the theorem. The second one can be assumed additionally after a rescaling argument explained in Lemma 2.2.3. We can now write

$$\Delta(g - g_0 - g_1) = O(\lambda). \tag{2.2.6}$$

This does not permit to show that the rest $\widetilde{g}_\lambda$ is a term of higher order in λ. This has consequences on the estimates of $\Box_{g_\lambda}\psi_\lambda$ as we see in the next calculations.

Parametrix for the scalar field

We compute:

$$\begin{aligned} \Box_{g_\lambda}\psi_\lambda = &(\Box_{g_\lambda} - \Box_{g_0})\psi_0 - \frac{1}{\lambda}\sum_{\mathbf{A}} g_\lambda^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta u_{\mathbf{A}} F_{\mathbf{A}}\cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \\ &-\sum_{\mathbf{A}}(2g_\lambda^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta F_{\mathbf{A}} + \Box_{g_\lambda} u_{\mathbf{A}} F_{\mathbf{A}})\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \lambda\sum_{\mathbf{A}}\Box_{g_\lambda} F_{\mathbf{A}}\cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \Box_{g_\lambda}\widetilde{\psi}_\lambda. \end{aligned}$$

Let us study the term with a factor $\frac{1}{\lambda}$. We recall that $u_{\mathbf{A}}$ satisfies the eikonal equation with respect to the metric g_0. Therefore we can write

$$\frac{1}{\lambda}g_\lambda^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta u_{\mathbf{A}} = \frac{1}{\lambda}(g_\lambda^{\alpha\beta} - g_0^{\alpha\beta})\partial_\alpha u_{\mathbf{A}}\partial_\beta u_{\mathbf{A}} = O(1),$$

where we have used (2.2.6). This is not sufficient to show that $\widetilde{\psi}_\lambda$, in the ansatz (2.2.4) is of higher order in λ ! The technique we used in [38] is to write an ansatz for the $O(\lambda^2)$ terms in ψ_λ. More precisely we write

$$\begin{aligned} \psi_\lambda = &\psi_0 + \sum_{\mathbf{A}}\lambda F_{\mathbf{A}}\cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}}\lambda^2\widetilde{F}_{\mathbf{A}}\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \\ &+\sum_{\mathbf{A}}\lambda^2\widetilde{F}_{\mathbf{A}}^{(2)}\cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}}\lambda^2\widetilde{F}_{\mathbf{A}}^{(3)}\sin\left(\frac{3u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}_\lambda. \end{aligned} \tag{2.2.7}$$

where $\widetilde{F}_{\mathbf{A}}$, $\widetilde{F}^{(2)}_{\mathbf{A}}$ and $\widetilde{F}^{(3)}_{\mathbf{A}}$ are carefully chosen to deal with the term

$$\frac{1}{\lambda}(g_\lambda^{\alpha\beta} - g_0^{\alpha\beta}) \sum_{\mathbf{A}} \partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{A}} F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right).$$

The presence of the oscillating terms in the metric, from the ansatz (2.2.5) leads to the creation of harmonics. Let us note that the terms of order one in λ that oscillate in a non null direction can be "integrated" with a better behaviour in λ, in the same way than for the elliptic equation.

Going back to the metric coefficient, we can check that in the right-hand side of $\Delta(g_\lambda - g_0 - g_1)$ the term of order one in λ are oscillating at a frequency proportional to λ. Consequently, we can find a high frequency ansatz g_2, of order λ^3 such that

$$\Delta(g_\lambda - g_0 - g_1 - g_2) = O(\lambda^2).$$

We will write $g = g_0 + g_1 + g_2 + g_3$, where g_3 is a remainder whose H^2 norm is of size λ^2. This, together with (2.2.7) is the final ansatz we take : the rest of the proof is a bootstrap argument on the remainders $\mathcal{E}_\lambda$ and g_3, together with $\tilde{F}_{\mathbf{A}}$ for which we have a transport equation that is coupled with the wave equation for $\mathcal{E}_\lambda$ and the elliptic equations for g_3..

Dealing with the time derivative of the metric coefficients

An important challenge to face is that the ∂_t derivative of the metric obeys worse estimates compared to the spatial derivatives. This is because the metric components solve elliptic equations on a spatial slice, and controlling their ∂_t derivative requires differentiating the equation. Fortunately, one can handle the situation using the structure of the Einstein equations! We highlight a few points below:

1. Hidden in the time evolution is the propagation of the maximality of the hypersurfaces. This allows us to rewrite the ∂_t derivative of a metric component (more precisely, γ, introduced in Section 2.1) in terms of spatial derivatives of another metric component (precisely, β, again introduced in Section 2.1). Hence, such a term is better than expected.

2. Importantly, using in particular the above observation, one shows that the two uncontrollable ∂_t-derivative error terms, one involving $\partial_t g_3$ and one involving $\partial_t^2 \widetilde{F}_{\mathbf{A}}$ ($\widetilde{F}_{\mathbf{A}}$ is coupled to g_3 via a transport equation) cancel. This apparently magical cancellation can be understood if instead of considering a parametrix of the form $\lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \lambda^2 \widetilde{F}_{\mathbf{A}} \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right)$ one uses a parametrix of the form $\lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}} + \widetilde{u}_{\mathbf{A}}}{\lambda}\right)$, where $u_{\mathbf{A}} + \widetilde{u}_{\mathbf{A}}$ is an approximate solution for the eikonal equation with the metric g. Then one could use Raychhaudury equation to show that $\Box_g(u_{\mathbf{A}} + \widetilde{u}_{\mathbf{A}})$ can be controlled by only one derivative of ψ and g.

3. Another type of error terms is of the form $\partial_t g_3$ multiplied by a low frequency term. Here, one can control the low frequency term using an integration by parts argument.

Rescaling of the phases and angular separation

At several points in the proof, we use that some terms involving the $u_{\mathbf{A}}$ are bounded away from zero, such as for instance $|\nabla(u_{\mathbf{A}} \pm u_{\mathbf{B}})|^2$. This can be ensured with a well chosen rescaling, using the following lemma.

Lemma 2.2.3. *Suppose $(g, \psi, F_{\mathbf{A}}, u_{\mathbf{A}})$ is a solution to (2.1.13) on $I \times \mathbb{R}^2$ for $I \subset \mathbb{R}$, in the sense of Theorem 2.1.5. For any set of positive constants $\{a_{\mathbf{A}}\}_{\mathbf{A}\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}_{>0}$, if we define*

$$F'_{\mathbf{A}} = a_{\mathbf{A}}^{-1} F_{\mathbf{A}}, \quad u'_{\mathbf{A}} = a_{\mathbf{A}} u_{\mathbf{A}},$$

then $(g, \psi, F'_{\mathbf{A}}, u'_{\mathbf{A}})$ is also a solution to (2.1.13).

By ordering I, and taking the $a_{\mathbf{A}_i}$ sufficiently large compared to the $a_{\mathbf{A}_j}$ for $j < i$, we can obtain for instance $|\nabla(a_{\mathbf{A}} u_{\mathbf{A}} \pm a_{\mathbf{B}} u_{\mathbf{B}})|^2 \geq 1$.

2.3 Construction of high-frequency space-times : from null dusts to Vlasov

In a work in progress with Jonathan Luk, we construct, in the $(2+1)$-dimensional space $[0,1]\times\mathbb{R}^2$ a sequence of solutions (g_i, ψ_i, ϖ_i) of

$$\begin{cases} \Box_g \psi + \frac{1}{2} e^{-4\psi} g^{-1}(d\varpi, d\varpi) = 0, \\ \Box_g \varpi - 4 g^{-1}(d\varpi, d\psi) = 0, \\ R_{\mu\nu}(g) = 2\partial_\mu\psi\partial_\nu\psi + \frac{1}{2} e^{-4\psi}\partial_\mu\varpi\partial_\nu\varpi \end{cases} \tag{2.3.1}$$

which converges to (g_0, ψ_0, ϖ_0), where , $(g_0, \psi_0, \varpi_0, f(\omega), u(\omega))$ is a solution of

$$\begin{cases} R_{\mu\nu}(g) = 2\partial_\mu\psi\partial_\nu\psi + \frac{1}{2} e^{-4\psi}\partial_\mu\varpi\partial_\nu\varpi + \int_{\mathbb{S}^1} f^2(t,x,\omega)\partial_\mu u(t,x,\omega)\partial_\nu u(t,x,\omega) dm(\omega), \\ \Box_g \psi + \frac{1}{2} e^{-4\psi} g^{-1}(d\varpi, d\varpi) = 0, \\ \Box_g \varpi - 4 g^{-1}(d\varpi, d\psi) = 0, \\ 2(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta f + (\Box_g u) f = 0 \quad \forall \omega \in \mathbb{S}^1, \\ (g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \quad u\big|_{\{t=0\}} = x\cdot\omega, \, \partial_t u\big|_{\{t=0\}} > 0, \quad \forall \varpi \in \mathbb{S}^1. \end{cases} \tag{2.3.2}$$

where $dm(\omega)$ is a probability measure on $\mathbb{S}^1$. In the sequel, we will note $U = (\psi, \varpi)$ and $\partial_\alpha U \cdot \partial_\beta U = \partial_\alpha\psi\partial_\beta\psi + \frac{1}{4} e^{-4\psi}\partial_\alpha\varpi\partial_\beta\varpi$.

Here, in (2.3.2), we have chosen a specific (and somewhat non-standard) parametrization of the cotangent bundle. Notice that the systems (2.3.1) and (2.3.2) in $(2+1)$-dimensions arise, respectively, as reductions of the Einstein vacuum equations and the Einstein–massless Vlasov system in $(3+1)$ dimensions under $\mathbb{U}(1)$ symmetry.

A rough statement of our theorem is the following

Theorem 2.3.1. *Let (g_0, U_0, f, u) be a sufficiently small and sufficiently regular local-in-time asymptotically conic solution to (2.3.2) such that*

- *The initial hypersurface is maximal;*
- *The vector field $g^{\alpha\beta}\partial_\beta u(\omega)$ is everywhere non zero;*
- *$g^{\alpha\beta}\partial_\alpha u(x,t,\omega)\partial_\beta u(x,t,\omega') \neq 0$ for all $\omega \neq \omega'$;*
- *A genericity condition holds.*

Then (g_0, U_0) can be weakly approximated by a sequence of solutions (g_i, U_i) to (2.3.1), i.e., in a suitable coordinate system (in fact in elliptic gauge); as $i \to \infty$, $(g_i, U_i) \to (g_0, U_0)$ uniformly on compact sets and the derivatives $(\partial g_i, \partial U_i) \rightharpoonup (\partial g_0, \partial U_0)$ weakly in L^2 (for each component).

The strategy of proof is the following

- First, we approximate the solution (g_0, U_0, f, u) by a sequence of solutions to Einstein Null dust (g^n, U^n, F^n_A, u^n_A).
- Second we approximate $(g^n, U^n, F^n_A, u^n_A))$ by a one parameter family of solutions to Einstein vacuum equations $(g^n_\lambda, U^n_\lambda)$. For this part, the proof of [38] has to be adapted to track down the dependency on n, the number of dusts : we can not assume that ε, the size of (g_0, U_0, f, u), is small compared to $\frac{1}{n}$. However, we can, and we will assume that λ is small compared to some other function of n going to zero as n goes to infinity.

2.3.1 Approximation of a Vlasov field by null dusts

It is a well-known fact, for instance by using the Krein–Milman theorem, that the set of convex combinations of Dirac measures is weak-* dense in the set of all probability measures. We will use a particular construction of a weak-* approximating sequence : let m be a probability measure on $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$. For all $n \in \mathbb{N}$, and $\mathbf{A} = 0, 1, \cdots, n-1$, we can find n separated points $\varpi^{(n)}_{\mathbf{A}} = \frac{\mathbf{A}}{2\pi n}$ on $\mathbb{S}^1$, and n coefficients $\sigma^{(n)}_{\mathbf{A}} = m\Big([\frac{\mathbf{A}}{2\pi n}, \frac{\mathbf{A}+1}{2\pi n})\Big)$ (so that $\sigma^{(n)}_{\mathbf{A}} \geq 0$ and $\sum_{\mathbf{A}=0}^{n-1} \sigma^{(n)}_{\mathbf{A}} = 1$) such that

$$\sum_{\mathbf{A}=0}^{n-1} \sigma^{(n)}_{\mathbf{A}} \delta_{\omega^n_{\mathbf{A}}} \overset{*}{\rightharpoonup} m, \tag{2.3.3}$$

in the weak-* topology as n tends to infinity. To approach (2.3.2) by n dusts, we consider the initial data for (2.3.2) $(\psi, \partial_t\psi, F^\psi(\omega), F^\varpi(\omega))$. We select the n parameters $\omega^N_{\mathbf{A}}$ and $\alpha^N_{\mathbf{A}}$ given by the claim and solve the coupled system

$$\begin{cases} R_{\mu\nu}(g) = 2\langle \partial_\mu U, \partial_\nu U\rangle + \sum_{\mathbf{A}} \alpha_{\mathbf{A}}((F^\psi_{\mathbf{A}})^2 + \frac{e^{-4\psi}}{4}(F^\varpi_{\mathbf{A}})^2)\partial_\mu u_{\mathbf{A}} \partial_\nu u_{\mathbf{A}}, \\ \Box_g \psi + \frac{1}{2} e^{-4\psi} g^{-1}(d\varpi, d\varpi) = 0, \\ \Box_g \varpi - 4 g^{-1}(d\varpi, d\psi) = 0, \\ 2(g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}} \partial_\beta F^\psi_{\mathbf{A}} + (\Box_g u_{\mathbf{A}}) F^\psi_{\mathbf{A}} + e^{-4\psi}(g^{-1})^{\alpha\beta}\partial_\alpha \varpi \partial_\beta u_{\mathbf{A}} F^\varpi_{\mathbf{A}} = 0, \\ 2(g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}} \partial_\beta F^\varpi_{\mathbf{A}} + (\Box_g u_{\mathbf{A}}) F^\varpi_{\mathbf{A}} - 4(g^{-1})^{\alpha\beta}\partial_\alpha \varpi \partial_\beta u_{\mathbf{A}} F^\psi_{\mathbf{A}} - 4(g^{-1})^{\alpha\beta}\partial_\alpha \psi \partial_\beta u_{\mathbf{A}} F^\varpi_{\mathbf{A}} = 0, \\ (g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}} \partial_\beta u_{\mathbf{A}} = 0. \end{cases} \tag{2.3.4}$$

where we have dropped the subscript n and noted $u_{\mathbf{A}} = u(\omega_{\mathbf{A}})$ and $F^{\psi,\varpi}_{\mathbf{A}} = F^{\psi,\varpi}(\omega_{\mathbf{A}})$. We have introduced a decomposition of the density. If we note $F^2_{\mathbf{A}} = (F^\psi_{\mathbf{A}})^2 + \frac{e^{-4\psi}}{4}(F^\varpi_{\mathbf{A}})^2$, we can check that it satisfies the transport equation

$$2(g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}} \partial_\beta F_{\mathbf{A}} + (\Box_g u_{\mathbf{A}}) F_{\mathbf{A}} = 0.$$

The importance of this decomposition will become clear in the next section.

We consider solutions with high regularity. Using a compactness argument, we can show the convergence of a subsequence toward a solution of (2.3.2).

2.3.2 Parametrix for ψ and ϖ

In order to approach a solution of (2.3.4) by a solution to Einstein vacuum equations, we show the existence of solutions to (2.1.2) of the form

$$\psi_\lambda = \psi_0 + \sum_{\mathbf{A}} \lambda \sigma_{\mathbf{A}} F^{\psi}_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}^{\psi}, \quad \varpi_\lambda = \varpi_0 + \sum_{\mathbf{A}} \lambda \sigma_{\mathbf{A}} F^{\varpi}_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}^{\varpi}. \tag{2.3.5}$$

$$g_\lambda = g_0 + \widetilde{g}_\lambda,$$

where $\mathcal{E}^{\psi,\varpi}, \widetilde{g}_\lambda$ are $O(\lambda^2)$.

The construction is similar to the one explained in Section 2.2.2, so we only point out the new difficulties

- There is already a difficulty to obtain the local existence, on a time interval depending on λ, of an ansatz of the form (2.3.5). We have

$$\partial\psi = \partial\psi_0 + \sum_{\mathbf{A}} \sigma_{\mathbf{A}} F^{\psi}_{\mathbf{A}} \partial u_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + O(\lambda),$$

  with $\sum \sigma_{\mathbf{A}}^2 = 1$. If for instance $\sigma_{\mathbf{A}} = \frac{1}{N^{\frac{1}{2}}}$, which is the case when m is the uniform measure on $\mathbb{S}^1$, then $\|\partial\psi\|_{L^\infty}$ is only bounded by $\varepsilon N^{\frac{1}{2}}$. This is a problem to apply the well posedness result of Theorem 2.1.5, and is why we need to use Theorem 2.1.6 instead. Indeed, we can show that the L^4 norm of $\partial\psi$ is of size ε. To see this, we have to develop the product of the sum in the L^4 norm. The diagonal terms are summable, and the non diagonal terms are oscillating: with an integration by part, one can obtain an additional power of λ, which will absorb the growth in N.

- More generally, a lot of terms in our construction can only be estimated with large constants depending on n. We need to track this dependency and adapt the bootstrap with a Gronwall type of argument : the remaining terms will only be estimated by $\lambda^2 e^{A(n)t}$. This is something possible because the terms involving product of the remainder are not the main terms and they can be estimated by higher powers of λ.

- The presence of the right-hand side in the wave equations for ψ and ϖ leads us to adapt the construction of our ansatz. A first adaptation is the coupled system satisfied by $F^{\psi}_{\mathbf{A}}$ and $F^{\varpi}_{\mathbf{A}}$, which is needed for the zero order terms to vanish. Similar adaptations are also necessary at higher order. Importantly, thanks to the null form, no harmonics are created at the first order.

2.4 Burnett's conjecture

In [39], we prove Burnett's conjecture when the metrics admit a $\mathbb{U}(1)$ symmetry and obey an elliptic gauge condition. A rough statement of our theorem is the following

Theorem 2.4.1. *Let $h_n = (g_n, \psi_n, \varpi_n)$ be a sequence of solutions of* (2.1.2) *in elliptic gauge, such that there exists $h_0 = (g_0, \psi_0, \varpi_0)$, smooth, also in elliptic gauge, with*

1. *$h_n \to h_0$ uniformly on compact sets,*

2. $\partial h_n \rightharpoonup \partial h_0$ *in* $L^{p_0}_{loc}$ *with* $p_0 > \frac{8}{3}$,

Then $T^{eff}_{\mu\nu}$*, which is the Einstein tensor of* h_0*, has the form of the stress-energy tensor of a massless Vlasov field. In addition if we have*

3. *for all compact* K *there exists* $\lambda_n \to 0$ *such that* $\sum_{k=0}^{4} \lambda_n^{k-1} \|\partial^k (h_n - h_0)\|_{L^\infty_{loc}} \lesssim C$

then Einstein-Vlasov equations are satisfied in the sense of measures.

Let us make some first comments on our theorem.

- The assumptions we imposed in the second part of the theorem are satisfied by the solutions constructed in [38]. However, they turned out to be unnecessary strong : they have been subsequently relaxed by Guerra and Teixeira da Costa in [30].
- We have to include the possibility of obtaining only measure solutions to Vlasov equations. Indeed the effective stress-energy tensor we obtain in [38] corresponds to a sum of null dusts: this is a Vlasov field where the density is a sum of Dirac measure in frequency space. Also, the tools we use in the proof of the Theorem 2.4.1 are the microlocal defect measures introduced by Gérard [26] and Tartar [65] so our result can be then naturally expressed in term of measures.

The following definition gives the precise form of the solution obtained in Theorem 2.4.1.

Definition 2.4.2 (Radially-averaged measure solutions for the restricted Einstein–massless Vlasov system in $\mathbb{U}(1)$ symmetry)**.** *Let* $({}^{(4)}\mathcal{M}, {}^{(4)}g)$ *be a* $(4+1)$*-dimensional* C^2 *Lorentzian manifold which is* $\mathbb{U}(1)$ *symmetric as in* (2.1.1)*, i.e. the metric takes the form*

$$ {}^{(4)}g = e^{-2\psi} g + e^{2\psi} (\mathrm{d}x^3 + \mathfrak{A}_\alpha \mathrm{d}x^\alpha)^2, $$

for g*,* ψ*,* $\mathfrak{A}$ *independent of* x^3*. Let* $\mathrm{d}\nu$ *be a non-negative finite Radon measure on* $S^*\mathcal{M}$*.*

We say that $({}^{(4)}\mathcal{M}, {}^{(4)}g, \mathrm{d}\nu)$ *is a radially-averaged measure solution for the restricted Einstein–massless Vlasov system in* $\mathbb{U}(1)$ *symmetry if*

1. *the following equations are satisfied:*

$$ \begin{cases} \Box_g \psi + \frac{1}{2} e^{-4\psi} g^{-1}(\mathrm{d}\varpi, \mathrm{d}\varpi) = 0, \\ \Box_g \varpi - 4 g^{-1}(\mathrm{d}\varpi, \mathrm{d}\psi) = 0, \\ \int_{\mathcal{M}} \mathrm{R}(g)(Y,Y)\, \mathrm{dVol}_g = \int_{\mathcal{M}} [2(Y\psi)^2 + \frac{1}{2} e^{-4\psi}(Y\varpi)^2]\, \mathrm{dVol}_g + \int_{S^*\mathcal{M}} \langle \xi, Y \rangle^2 \frac{\mathrm{d}\nu}{|\xi|^2}, \end{cases} \tag{2.4.1} $$

for every C^∞_c *vector field* Y*, where* ϖ *relates to* $\mathfrak{A}_\alpha$ *via* (2.1.3) *and where* $S^*\mathcal{M}$ *is the cosphere bundle given by* $S^*\mathcal{M} = (T^*\mathcal{M} \setminus \{0\}) / \sim$*, where* $(x, \xi) \sim (y, \eta)$ *if and only if* $x = y$ *and* $\xi = \lambda \eta$ *for some* $\lambda > 0$*;*

2. $\mathrm{d}\nu$ *is supported on the zero mass shell in the sense that for all* $f \in C_c(\mathcal{M})$*,*

$$ \int_{S^*\mathcal{M}} f(x) (g^{-1})^{\alpha\beta} \xi_\alpha \xi_\beta \frac{\mathrm{d}\nu}{|\xi|^2} = 0; $$

3. *For any* C^1 *function* $\widetilde{a} : T^*\mathcal{M} \to \mathbb{R}$ *which is homogeneous of degree* 1 *in* ξ*,*

$$ \int_{S^*\mathcal{M}} \left((g^{-1})^{\alpha\beta} \xi_\beta \partial_{x^\alpha} \widetilde{a} - \frac{1}{2} (\partial_\mu g^{-1})^{\alpha\beta} \xi_\alpha \xi_\beta \partial_{\xi_\mu} \widetilde{a} \right) \frac{\mathrm{d}\nu}{|\xi|^2} = 0. \tag{2.4.2} $$

Before giving some ideas of the proof, we recall some notions about microlocal defect measures in Section 2.4.1. Then in Section 2.4.2 we will sketch the proof of the first part of Theorem 2.4.1. Finally, in Section 2.4.3 we sketch the proof of the second part of Theorem 2.4.1, that is the transport equation satisfied by the density in the effective stress-energy tensor of the Vlasov field. This is by far the most delicate part of [39].

2.4.1 Preliminaries on microlocal defect measures

The microlocal defect measure is a measure on the cosphere bundle which identifies the "region in phase space" for which strong convergence fails. An important property of microlocal defect measures, especially relevant for our problem, is that microlocal defect measures arising from (approximate) solutions to hyperbolic equations themselves satisfy some transport equations.

Let $\{u_n\}$ be a sequence of functions $\Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ is open, which converges *weakly* in $L^2(\Omega)$ to a function u. In general, after passing to a subsequence, $|u_n|^2 - |u|^2$ converges to a non-zero measure. The failure of the convergence $|u_n|^2 \to |u|^2$ can arise from concentrations or oscillations. The microlocal defect measure is a tool which captures both the position and the frequency of this failure of strong convergence.

For instance, if $u_n = n^{\frac{d}{2}}\chi(n(x-x_0))$ (with $\chi \in C_c^\infty$) so that $|u_n|^2$ concentrates to a delta measure, then the corresponding microlocal defect measure is given by $\delta_{x_0} \otimes \nu$, where δ_{x_0} is the spatial delta measure and ν is a uniform measure on the cotangent space. On the other hand, suppose $u_n(x) = \chi(x)\cos(n(x\cdot\omega))$ so that u_n oscillates in a particular frequency ω. Then the corresponding microlocal defect measure is $|\chi|^2 \mathrm{d}x \otimes \delta_{[\omega]}$, where $\delta_{[\omega]}$ is the delta measure concentrated at the (equivalent class of the) frequency ω. See [65] for further discussions.

Before introducing the precise definition of microlocal measures, we need to recall some objects of pseudo-differential calculus. In the rest of this section, we fix $k \in \mathbb{N}$ (which will be taken as $3 = 2+1$ in next sections). Denote by $T^*\mathbb{R}^k$ the cotangent bundle of $\mathbb{R}^k$ with coordinates $(x,\xi) \in \mathbb{R}^k \times \mathbb{R}^k$.

Definition 2.4.3. *1. For $m \in \mathbb{Z}$, define the symbol class*

$$S^m := \{a : T^*\mathbb{R}^k \to \mathbb{C} : a \in C^\infty, \forall \alpha, \beta, \exists C_{\alpha,\beta} > 0, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\}.$$

2. *Given a symbol $a \in S^m$, define the operator $\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^k)$ by*

$$(\mathrm{Op}(a)u)(x) := \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi.$$

We say that $A = mathrmOp(a)$ is a pseudo-differential operator of order m with symbol a. If moreover $a(x,\xi) = a_{\mathrm{prin}}(x,\xi)\chi(\xi) + a_{\mathrm{error}}$, where $a_{\mathrm{prin}}(x,\lambda\xi) = \lambda^m a(x,\xi)$ for all $\lambda > 0$, $\chi(\xi)$ is a cutoff defined by (2.1.2) and $a_{\mathrm{error}} \in S^{m-1}$, we say that a_{prin} is the principal symbol of A.

We now turn to the discussion of microlocal defect measures, following [26] (see also [65]). We first need some preliminary definitions.

Definition 2.4.4. *We say that $\mathrm{d}\mu$ is a non-negative $(N \times N)$-complex-matrix-valued Radon measure on $S^*\mathbb{R}^k$ if $\mathrm{d}\mu$ is a map $\mathrm{d}\mu : C_c(S^*\mathbb{R}^k) \to \mathbb{C}^{N\times N}$*

1. *obeying the estimate $\|\mathrm{d}\mu(\varphi)\| \le C_K \|\varphi\|_{C(K)}$ for every compact set $K \subset S^*\mathbb{R}^k$ (for some $C_K > 0$ depending on K), and*

2. *satisfying* $\mathrm{d}\mu(\varphi)$ *is a positive semi-definite Hermitian matrix whenever* φ *is a non-negative function.*

Definition 2.4.5. *Let* $\mathrm{d}\mu$ *be a non-negative* $(N \times N)$*-complex-matrix-valued Radon measure on* $S^*\mathbb{R}^k$ *and* $a\!\!\!/ : S^*\mathbb{R}^k \to \mathbb{C}^{N\times N}$ *be a continuous matrix-valued function on* $S^*\mathbb{R}^k$.

Define $\mathrm{tr}\,(a\!\!\!/(x,\xi)\,\mathrm{d}\mu)$ *to be the (scalar-valued) Radon measure on* $S^*\mathbb{R}^k$ *given by*

$$(\mathrm{tr}\,(a\!\!\!/(x,\xi)\,\mathrm{d}\mu))(\varphi) := \mathrm{tr}\,[a\!\!\!/(x,\xi)\cdot(\mathrm{d}\mu(\varphi))].$$

Theorem 2.4.6 (Existence of microlocal defect measures, Theorem 1 in [26])**.** *Let* $\{u_n\}_{n=1}^{+\infty} \in L^2(\mathbb{R}^k;\mathbb{C}^N)$ *be a bounded sequence such that* $u_n \rightharpoonup 0$ *weakly in* $L^2(\mathbb{R}^k;\mathbb{C}^N)$.

Then there exists a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ *and a non-negative* $(N\times N)$*-complex-matrix-valued Radon measure* $\mathrm{d}\mu$ *on* $S^*\mathbb{R}^k$ *such that for every* $\mathbb{C}^{N\times N}$*-valued pseudo-differential operator* $A\!\!\!/$ *of order* 0 *with principal symbol* $a\!\!\!/ \in C_c(S^*\mathbb{R}^k;\mathbb{C}^{N\times N})$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^k}\langle A\!\!\!/ u_{n_k}, u_{n_k}\rangle_{\mathbb{C}^N}\,\mathrm{d}x = \int_{S^*\mathbb{R}^k}\mathrm{tr}\,(a\!\!\!/(x,\xi)\,\mathrm{d}\mu).$$

The measure $\mathrm{d}\mu$ in Theorem 2.4.6 is called a *microlocal defect measure*. Following [26], if the conclusion of Theorem 2.4.6 holds for the whole sequence $\{u_n\}_{n=1}^{+\infty}$, we say that $\{u_n\}_{n=1}^{+\infty}$ is a *pure* sequence.

Theorem 2.4.7 (Localization of microlocal defect measures, Corollary 2.2 in [26])**.** *Let* $\{u_n\}$ *be a pure sequence of* $L^2(\mathbb{R}^k,\mathbb{C}^N)$, *of microlocal defect measure* $\mathrm{d}\mu$. *Let* P *be an* m*-th order differential operator with principal symbol* $p = \sum_{|\alpha|=m} a_\alpha(i\xi)^\alpha$ *for some smooth* $(N\times N)$*-matrices* a_α. *If* $\{Pu_n\}_{n=1}$ *is relatively compact in* $H^{-m}_{loc}(\mathbb{R}^k,\mathbb{C}^N)$, *then*

$$p\,\mathrm{d}\mu = 0.$$

2.4.2 The form of the effective stress-energy tensor

In this section, we sketch the proof of the first part of Theorem 2.4.1. Let $h_n = (g_n,\psi_n,\varpi_n)$ be a sequence of solutions of (2.1.2) in elliptic gauge which satisfies the conditions 1 and 2 of the theorem. By multiplying the sequence by some cut-off functions χ, we reduce the problem to compact sets. The sequences $\chi(\psi_n-\psi_0)$ and $\chi(\varpi_n-\varpi_0)$ are such that

$$\partial(\chi(\psi_n-\psi_0)) \rightharpoonup 0, \quad \partial(\chi(\varpi_n-\varpi_0)) \rightharpoonup 0,$$

weakly in $L^2(\mathbb{R}^{2+1})$. Therefore we can apply Theorem 2.4.6 and obtain the existence of Radon measures $\mathrm{d}\sigma^\psi_{\alpha\beta}$, $\mathrm{d}\sigma^\varpi_{\alpha\beta}$ and $\mathrm{d}\sigma^{\mathrm{cross}}_{\alpha\beta}$ on $S^*\mathbb{R}^{2+1}$ such that for any zeroth order (scalar) pseudo-differential operators $\{A^{\alpha\beta}\}_{\alpha,\beta=t,1,2}$ with principal symbols $\{a^{\alpha\beta}\}_{\alpha,\beta=t,1,2}$, the following holds up to a subsequence (which we do not relabel):

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2+1}}\partial_\alpha(\chi(\psi_n-\psi_0))A^{\alpha\beta}\partial_\beta(\chi(\psi_n-\psi_0))\,\mathrm{dVol}_{g_0} = \int_{S^*\mathbb{R}^{2+1}}a^{\alpha\beta}\,\mathrm{d}\sigma^\psi_{\alpha\beta},$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^{2+1}}\partial_\alpha(\chi(\varpi_n-\varpi_0))A^{\alpha\beta}\partial_\beta(\chi(\varpi_n-\varpi_0))\,\mathrm{dVol}_{g_0} = \int_{S^*\mathbb{R}^{2+1}}a^{\alpha\beta}\,\mathrm{d}\sigma^\varpi_{\alpha\beta},$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^{2+1}}\partial_\alpha(\chi(\psi_n-\psi_0))A^{\alpha\beta}\partial_\beta(\chi(\varpi_n-\varpi_0))\,\mathrm{dVol}_{g_0} = \int_{S^*\mathbb{R}^{2+1}}a^{\alpha\beta}\,\mathrm{d}\sigma^{\mathrm{cross}}_{\alpha\beta},$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^{2+1}}\partial_\alpha(\chi(\varpi_n-\varpi_0))A^{\alpha\beta}\partial_\beta(\chi(\psi_n-\psi_0))\,\mathrm{dVol}_{g_0} = \int_{S^*\mathbb{R}^{2+1}}a^{\alpha\beta}\,(\mathrm{d}\sigma^{\mathrm{cross}})^*_{\alpha\beta},$$

where * denotes the Hermitian conjugate. Moreover, $\mathrm{d}\sigma^{\psi}_{\alpha\beta}$ and $\mathrm{d}\sigma^{\varpi}_{\alpha\beta}$ are non-negative in the sense of Definition 2.4.4, and $\mathrm{d}\sigma^{\mathrm{cross}}_{\alpha\beta} = \mathrm{d}\sigma^{\mathrm{cross}}_{\beta\alpha}$. Since the presence of the cut-off function does not create real difficulties, we drop it in the rest of this presentation.

Using the fact that we are considering the microlocal defect measures associated to a sequence of derivatives, we have a specific form for $\mathrm{d}\sigma^{\psi}_{\alpha\beta}$ and $\mathrm{d}\sigma^{\varpi}_{\alpha\beta}$ (something similar could be obtain for $\mathrm{d}\sigma^{\mathrm{cross}}_{\alpha\beta}$ but we do not need it in [39].)

Proposition 2.4.8 (Microlocal defect measures are effectively given by $\mathrm{d}\nu^{\psi}$ and $\mathrm{d}\nu^{\varpi}$)**.** *There exist non-negative Radon measures $\mathrm{d}\nu^{\psi}$, $\mathrm{d}\nu^{\varpi}$ on $S^*\mathbb{R}^{2+1}$ such that*

$$\mathrm{d}\sigma^{\psi}_{\alpha\beta} = \frac{\xi_\alpha\xi_\beta}{|\xi|^2}\mathrm{d}\nu^{\psi}, \quad \mathrm{d}\sigma^{\varpi}_{\alpha\beta} = \frac{\xi_\alpha\xi_\beta}{|\xi|^2}\mathrm{d}\nu^{\varpi},$$

where $|\xi|^2 := \sum_{\mu=0}^{2} |\xi_\mu|^2$.

We are now ready to pass to the weak limit in (2.1.2). In the wave-map system

$$\begin{cases} \Box_{g_n}\psi_n + \frac{1}{2}e^{-4\psi_n}g^{-1}(\mathrm{d}\varpi_n, \mathrm{d}\varpi_n) = 0, \\ \Box_{g_n}\varpi_n - 4g_n^{-1}(\mathrm{d}\varpi_n, \mathrm{d}\psi_n) = 0, \end{cases}$$

the term in divergence form and the null forms pass to the weak limit, the system satisfied by (ψ_0, ϖ_0) is the same. Let us now consider the equation

$$\mathrm{Ric}_{\alpha\beta}(g_n) = 2\partial_\alpha\psi_n\partial_\beta\psi_n + \frac{1}{2}e^{-4\psi_n}\partial_\alpha\varpi_n\partial_\beta\varpi_n. \tag{2.4.3}$$

The left hand side converges weakly to $\mathrm{Ric}_{\alpha\beta}(g_0)$. To see this, we refer to the expression of the Ricci tensor in elliptic gauge, given by (2.1.7) to (2.1.12). The quadratic terms are of the form H^2, where H is the second fundamental form, or $\nabla\gamma\nabla N$. It turns out that the quantities H and γ have additional properties: they satisfy both an elliptic equation, whose right-hand side, of the form $(\partial h_n)^2$ is bounded in $L^{\frac{p_0}{2}}$, and an equation involving ∂_t. This allows to show that H_n and $\nabla\gamma_n$ are bouded in $W^{1,\frac{p_0}{2}}(\mathbb{R}^{3+1})$, which compactly embeds in L^2 with our choice of $p_0 > \frac{8}{3}$. Consequently H_n and $\nabla\gamma_n$ have strong L^2 limits. The factor ∇N_n is only bounded in L^2, but it is sufficient to show the weak convergence of $\nabla\gamma_n\nabla N_n$.

The weak limit of the right-hand side of (2.4.3) can be expressed in term of the microlocal defect measures associated to ψ_n and ϖ_n. Let

$$\mathrm{d}\nu := 2\mathrm{d}\nu^{\psi} + \frac{1}{2}e^{-4\psi_0}\,\mathrm{d}\nu^{\varpi}. \tag{2.4.4}$$

Then the limiting metric g_0 satisfies, for every vector field $Y \in C^\infty_c(\Omega)$.

$$\int_{\mathbb{R}^{2+1}} \mathrm{Ric}(g_0)(Y,Y)\,\mathrm{dVol}_{g_0} = \int_{\mathbb{R}^{2+1}} \left(2(Y\psi_0)^2 + \frac{1}{2}e^{-4\psi_0}(Y\varpi_0)^2\right)\mathrm{dVol}_{g_0} + \int_{S^*\mathbb{R}^{2+1}} (Y^\alpha\xi_\alpha)^2\,\mathrm{d}\nu.$$

The fact that the measure $\mathrm{d}\nu$ is supported on the set $\{(x,\xi) \in S^*\mathcal{M} : g_0^{-1}(\xi,\xi) = 0\}$ can easily be obtained from Theorem 2.4.7 and the fact that $\Box_{g_0}(\psi_n - \psi_0)$ and $\Box_{g_0}(\varpi_n - \varpi_0)$ are compact in H^{-1}.

2.4.3 The transport equation for the effective density

Microlocal measures arising from solutions to linear equations satisfy the massless Vlasov equation (see for instance [24]). In the case of a wave equation on Minkowski $\Box\phi_n = f_n$, where $\partial\phi_n \rightharpoonup 0$ in L^2 and f_n is compact in H^1, this can be seen by an easy computation of a commutator. For a pseudo differential operator A of order 0, with real principal symbol, we note

$$I_n = \langle [A,\partial_t]\partial_t\phi_n, \partial_t\phi_n\rangle - \langle [A,\partial_i]\delta^{ij}\partial_j\phi_n, \partial_t\phi_n\rangle + \langle [A,\partial_t]\delta^{ij}\partial_i\phi_n, \partial_j\phi_n\rangle - \langle [A,\partial_i]\delta^{ij}\partial_t\phi_n, \partial_j\phi_n\rangle.$$

On one side, we have

$$I_n \to 2\int_{S^*\mathbb{R}^{2+1}} (-\xi_0^2\partial_t a + \delta^{ij}\xi\xi_0\partial_j a))\frac{\mathrm{d}\nu}{|\xi|^2},$$

as $n\to\infty$, where $\mathrm{d}\nu$ is the defect measure associated to ϕ following the same steps as in the previous section. By replacing a by $\frac{\tilde{a}}{\xi_0}$, where $\tilde{a}$ is of order 1, we obtain the left-hand side of (2.4.2) in the particular case where g is the Minkowski metric. The quantity I_n can be also calculated by integration by parts. We have

$$I_n = \langle A\Box\phi_n, \partial_t\phi_n\rangle + \langle A\partial_t\phi_n, \Box\phi_n\rangle. \tag{2.4.5}$$

In the case where f_n is compact in L^2, we obtain an homogeneous transport equation for $d\mu$.

$$2\int_{S^*\mathbb{R}^{2+1}} (-\xi_0\partial_t\tilde{a} + \delta^{ij}\xi\partial_j\tilde{a})\frac{\mathrm{d}\nu}{|\xi|^2} = 0.$$

Their are a few difficulties to adapting this to the present problem which are described in the next paragraphs.

Trilinear compensated compactness for three waves

The right hand side of the equations for ψ_n and ϖ_n are not compact in L^2. However, they have a special structure which can be used to obtain trilinear compactness (see also [25] and [45]). Let us explain the phenomena without geometric setting. We consider three sequences of functions $\phi_n^{(i)}$, with $i = 1..3$ which are such that

$$\sum_{k=0}^{2}\lambda_n^k\|\partial^k\phi_n^{(i)}\|_{L^\infty} \lesssim \lambda_n, \qquad \|\Box\phi_n^{(i)}\|_{L^\infty} \lesssim 1,$$

with $\lambda_n \to 0$ as $n\to\infty$. We claim that we have for any pseudo-differential operator A of order 0,

$$\langle A(\partial_t\phi_n^{(1)}\partial_t\phi_n^{(2)} - \nabla\phi_n^{(1)}.\nabla\phi_n^{(2)}), \partial_t\phi_n^{(3)}\rangle \to 0.$$

To see this we note that

$$2(\partial_t\phi_n^{(1)}\partial_t\phi_n^{(2)} - \nabla\phi_n^{(1)}.\nabla\phi_n^{(2)}) = \Box(\phi^{(1)}\phi^{(2)}) - \phi^{(1)}\Box\phi^{(2)} - \phi^{(2)}\Box\phi^{(1)}.$$

Consequently, we can compute

$$\begin{aligned}\langle A(\partial_t\phi_n^{(1)}\partial_t\phi_n^{(2)} - \nabla\phi_n^{(1)}.\nabla\phi_n^{(2)}), \partial_t\phi_n^{(3)}\rangle =&\langle A\Box(\phi_n^{(1)}\phi_n^{(2)}), \partial_t\phi_n^{(3)}\rangle + O(\lambda_n)\\ =&-\langle A\partial_t(\phi_n^{(1)}\phi_n^{(2)}), \Box\phi_n^{(3)}\rangle + O(\lambda_n)\\ =&O(\lambda_n)\end{aligned}$$

This finishes the proof of the claim.

Elliptic-wave trilinear compensated compactness.

What has been described above can be adapted in a geometrical setting. However there are extra difficulties when the metric depends on the solution : we have to analyse the behaviour of trilinear terms of the form

$$\langle A(\partial(g_n - g_0)\partial\psi_n), \partial_t\psi_n\rangle.$$

If the term ∂g_n is a spacial derivative, it can be handled since the metric coefficients satisfy elliptic equations. We have in fact $\|\nabla(g_n - g_0)\|_{L^\infty} \leq \lambda_n^{\frac{1}{2}}$. On the contrary, a term involving $\partial_t g_n$ could lead to defect of compactness if it had no additional structure. Fortunately, the terms we consider have an additional structure. This structure comes

- either from the fact that a particular metric coefficient is involved (see Section 2.1 for the definitions of t), such as γ_n for which we have an additional equation involving a ∂_t derivative (see Section 2.1 for the definitions): in this case $\|\partial_t(\gamma_n - \gamma_0)\|_{L^\infty} \leq \lambda_n^{\frac{1}{2}}$;
- either from the commutator structure.

A term of the form $\partial_t N_n$ (see Section 2.1 for the definition of N_n), does not appear in the wave equation. However we have some terms of the following form to study, in which we will use the commutator structure:

$$I_n = \langle [A, N_n - N_0], \Delta\psi_n, \partial_t\psi_n\rangle.$$

where N_n is one of the metric coefficients. We can assume that A is simply a Fourier multiplier with symbol $a(x,\xi) = m(\xi)$ independent of x. This indeed captures the main difficulty. In this case, since ϕ_n is real-valued, we can also assume that m is even. Under these assumptions, we use Plancherel equality to rewrite

$$\begin{aligned} I_n &= \int_{\mathbb{R}^{2+1}} \partial_t\psi_n \delta^{ij}\{(N_n - N_0)A\partial^2_{ij}\psi_n - A[(N_n - N_0)\partial^2_{ij}\psi_n]\}\,\mathrm{d}x \\ &= \frac{i}{2}\int_{\mathbb{R}^{2+1}\times\mathbb{R}^{2+1}} (\xi_t|\eta_i|^2 + \eta_t|\xi_i|^2)(\widehat{N_n - N_0})(\eta - \xi)\widehat{\psi_n}(-\eta)\widehat{\psi_n}(\xi)(m(\xi) - m(\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{2.4.6}$$

where we decomposed ξ and η into their time and spatial parts: $\xi = (\xi_t, \xi_i), \eta = (\eta_t, \eta_i)$.

Roughly speaking $(\xi_t|\eta_i|^2 + \eta_t|\xi_i|^2)$ corresponds to three derivatives, and hence contributes to $O(\lambda_n^{-3})$ in size. This is just enough to show that I_n is bounded. To deduce that I_n actually tends to 0, observe

- our main enemy is when $N_n - N_0$ has high-frequency in t, i.e. $|\eta_t - \xi_t|$ is large (since we have better estimates for spatial derivatives of N_n.
- we can gain with factors of $\xi_i - \eta_i$ (corresponding to spatial derivatives of $N_n - N_0$) or $\xi_t^2 - |\xi_i|^2$ or $\eta_t^2 - |\eta_i|^2$ (corresponding to $\Box_{g_0}$ acting on ψ_n).

Now the Fourier multiplier in I_n can be written as

$$\xi_t|\eta_i|^2 + \eta_t|\xi_i|^2 = \eta_t(\xi_i + \eta_i)(\xi_i - \eta_i) + |\eta_i|^2(\xi_t + \eta_t).$$

The first term contains a factor of $(\xi_i - \eta_i)$ which as mentioned above corresponds to a spatial derivative of N_n and behaves better. For the second term, we rewrite

$$|\eta_i|^2(\xi_t + \eta_t) = |\eta_i|^2\frac{\xi_t^2 - \eta_t^2}{\xi_t - \eta_t} = |\eta_i|^2\frac{\xi_t^2 - |\xi_i|^2}{\xi_t - \eta_t} + |\eta_i|^2\frac{|\eta_i|^2 - \eta_t^2}{\xi_t - \eta_t} + |\eta_i|^2\frac{(\xi_i + \eta_i)\cdot(\xi_i - \eta_i)}{\xi_t - \eta_t}.$$

When $\xi_t - \eta_t$ is large, we can make use of the gain in $\xi_t^2 - |\xi_i|^2$, $|\eta_i|^2 - \eta_t^2$ or $(\xi_i - \eta_i)$ to conclude that this term behaves better than expected.

The wave map structure

In the wave equation satisfied by $\psi_n - \psi_0$ we have linear terms of the form

$$e^{-4\psi_0} g_0^{-1}(d(\varpi_n - \varpi_0), d\varpi_0)$$

and in the equation for $\varpi_n - \varpi_0$,

$$-2g_0^{-1}(d(\varpi_n - \varpi_0), d\psi_0) - 2g_0^{-1}(d\varpi_0, d(\psi_n - \psi_0)).$$

To obtain the transport equation for $\mathrm{d}\nu^\psi$ and $\mathrm{d}\nu^\varpi$ we have to pass to the limit in expressions of the form (2.4.5). Therefore, these linear terms induce a contribution in the transport equations: for instance

$$\langle A(e^{-4\psi_0} g_0^{-1}(d(\varpi_n - \varpi_0), d\varpi_0)), \partial_t \psi_n \rangle \to \int a(x, \xi) e^{-4\psi_0} g_0^{\alpha\beta} \xi_\beta \xi_0 \mathrm{d}\sigma_{\beta t}^{cross}.$$

It turns out that these contributions cancel when one considers the transport equation for the measure $\mathrm{d}\nu$ defined by (2.4.4). The same phenomena was observed in the construction of Section 2.3.

2.5 Perspectives

A first natural perspective is of course to remove the $\mathbb{U}(1)$ symmetry assumption. In [58], Luk and Rodnianski worked without symmetry, but in a double null gauge and for solutions which are regular in the angular directions. This setting is adapted to the study of two high-frequency waves propagating in orthogonal directions but cannot be used in more general cases. To study the interaction of more than two oscillating waves, a study in wave coordinates seems more adapted. In his thesis, Arthur Touati has initiated the construction of a high frequency wave propagating in one direction, in a setting which should allow to study the superposition of several waves. The structure of Einstein equations in wave coordinates seems also well adapted to the study of Burnett's conjecture itself.

Another perspective is to consider high-frequency limits in the case of non vacuum Einstein equations, and in particular for Euler-Einstein equations. This is of particular interest since it is a model actually considered in cosmology, and used for instance in [21].

Chapter 3

Einstein equations on product spaces with compact directions

The idea of considering space-times in more than 4 dimensions comes from the work of Kaluza in 1921 [46] and Klein in 1926 [54]. They consider a 5 dimensional Lorentzian manifolds $(\mathcal{M}\times\mathbb{U}(1),g)$, symmetric in the $\mathbb{U}(1)$ direction. Such a metric g can be decomposed in the following way :

$$^{(5)}g = g_{\alpha\beta}dx^\alpha dx^\beta + \Phi(dy + A_\alpha dx^\alpha)^2.$$

Einstein vacuum equations $^{(5)}R_{\mu\nu} = 0$ then imply Einstein Maxwell scalar field equations on $\mathcal{M}$ for $\tilde{g} = \Phi^{\frac{1}{2}}g$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $f = \ln(\Phi)$

$$\begin{array}{l} \tilde{R}_{\alpha\beta} = \frac{1}{2}F_\beta{}^\mu F_{\alpha\mu} - \frac{1}{8}g_{\alpha\beta}F_{\lambda\mu}F^{\lambda\mu} + 6\partial^\alpha f\partial_\alpha f \\ D^\alpha\partial_\alpha f = -\frac{1}{2}\partial^\alpha f\partial_\alpha f + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} \\ D^\alpha(e^{2f}F_{\alpha\beta}) = 0. \end{array} \tag{3.0.1}$$

This symmetry reduction is similar to the $\mathbb{U}(1)$ symmetry in $3+1$ dimension, presented in Chapter 2. The theory has been later extended by Kerner (1968) [47] to non Abelian isometry groups. If $(\tilde{\mathcal{M}},\tilde{g})$ is a $4+d$ dimensional Lorentzian manifold, invariant under the action of a compact Lie group G then Einstein vacuum equations in $4+d$ dimensions yield Einstein-Yang-Mills equations in 4 dimensions (see also Appendix 8 in [11] for the calculations). The introduction of additional compact directions is now present in many modern Physics theories such as supersymmetry and supergravity [9].

If we think that Einstein vacuum equation on $(\mathcal{M}\times K)$, with $\mathcal{M}$ a $3+1$ dimensional manifold and K a compact manifold, is an interesting model, then it is important to understand the effect of relaxing the symmetry assumption with respect to variables in K (see for instance [4]). Let us consider the simpler case $\mathbb{R}^{3+1}\times\mathbb{U}^1$ with $g = -dt^2 + dx^2 + R^2dy^2$, and the equation $\Box_g\phi = 0$. A Fourier series expansion of ϕ with respect to $\mathbb{U}(1)$, $\phi = \sum\phi_k$ yields the equations

$$\Box_{t,x}\phi_k - \frac{k^2}{R^2}\phi_k = 0.$$

A physical assumption is that R is very small (smaller than Planck length). Then the non zero modes ϕ_k are of very high energy, inaccessible to observations.

Mathematically, the question remains to study the effect of additional compact directions in the dynamics. In term of stability, let us briefly describe two results

- In [67], Wyatt proved the stability of Kaluza Klein spacetimes where only perturbations depending on the non-compact coordinates are considered, using tools developed by Lindblad and Rodninaski [57]. In other words, she proves the stability of the trivial solution $(g, F, \Phi) = (m, 0, 0)$, where m is the Minkowski metric for the system (3.0.1).
- In [2], Andersson, Blue, Wyatt, and Yau studied more general spacetimes with supersymmetric compactifications $\mathcal{M} = \mathbb{R}^{1+n} \times K$. The space-times $\mathcal{M}$ are equipped with the metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + k$$

  where $\eta_{\mathbb{R}^{1+n}}$ is the Minkowski metric in $\mathbb{R}^{1+n}$ and k is such that (K, k) is a compact Ricci-flat Riemannian manifold. A global stability result is proved under the assumption $n \geq 9$ and for Cauchy data that are Schwarzschild outside a compact domain.

In this chapter, I will present two results on this topic. The first one concerns the constraint equations for Einstein equations posed on $\mathbb{R}^{1+n} \times \mathbb{T}^d$. This is a fundamental ingredient in order to apply Choquet-Bruhat theorem and show the local well-posedness for Einstein equations on product spaces. The result, presented in Section 3.1 is a joint work with Caterina Vâlcu. The second result concerns a toy model for the stability of the trivial solution $g = m + dy^2$ on $\mathbb{R}^{3+1} \times \mathbb{S}^1$, which is just Minkowski solution on $\mathbb{R}^{4+1}$ compactified in one direction. This is a joint work with Annalaura Stingo, and the object of Section 3.2.

3.1 Constraint equations on product space

In [41], we address the resolution of the constraints for Einstein equations on the product manifold $\mathbb{R}^{1+n} \times \mathbb{T}^m$. We denote by $d = n + m$ the total dimension of the initial hypersurface $M = \mathbb{R}^n \times \mathbb{T}^m$, and $2^* = \frac{2d}{d-2}$. We recall from Section 1.2 that the initial data $(\Sigma, \bar{g}, K)$ for Einstein equations can not be chosen arbitrarily and have to solve the so called constraint equations. Using the conformal change $\hat{g} = \varphi^{2^*-2} g$, $\varphi > 0$, these equations take the form

$$\begin{aligned} \frac{4(d-1)}{d-2}\Delta_g\varphi + R_g\varphi &= -\frac{d-1}{d}\tau^2\varphi^{2^*-1} + \frac{|U + \mathcal{L}_g W|_g^2}{\varphi^{2^*+1}} \\ \overrightarrow{\Delta}_g W &= \frac{d-1}{d}\varphi^{2^*} d\tau \end{aligned} \tag{3.1.1}$$

where $\tau = tr_{\hat{g}}\hat{K}$, U is a symmetric 2-tensor, and we have denoted $\mathcal{L}_g$ the conformal Killing operator

$$(\mathcal{L}_g W)_{\mu\nu} = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{d} div_g W g_{ij}, \quad \forall \mu, \nu \in \overline{1, d}$$

and $\overrightarrow{\Delta}_g$ the vectorial conformal Laplacian

$$(\overrightarrow{\Delta}_g W)^\nu = -\nabla_\mu \left(\nabla^\mu W^\nu + \nabla^\nu W^\mu - \frac{2}{d} g^{\mu\nu} \nabla_k W^k \right), \quad \forall \mu, \nu \in \overline{1, d}$$

where ∇ is the Levi-Civitta connection associated to g.

3.1.1 Weighted Sobolev spaces

Before stating our result on the resolution of (3.1.1), we introduce functional spaces. In [60], Mc Owen showed that the Laplacian on $\mathbb{R}^n$ had good invertibility properties on the weighted spaces $W^p_{s,\delta}(\mathbb{R}^n)$ defined by

$$\|f\|^p_{W^p_{s,\delta}(\mathbb{R}^n)} = \sum_{0\leq|\beta|\leq s}\int_{\mathbb{R}^n}|\partial_x^\beta f|^p\langle x\rangle^{p(\delta+|\beta|)}\,dx$$

We introduce new weighted Sobolev spaces, adapted to the behaviour of the Laplacian on $\mathbb{R}^n\times\mathbb{T}^m$

$$\Delta = -\sum_{i=1}^n \partial^2_{x_i} - \sum_{j=1}^m \partial^2_{\theta_j}$$

where $(x_i)_{i\in\overline{1,n}}$ are the coordinates of $\mathbb{R}^n$ and $(\theta_j)_{j\in\overline{1,m}}$ are the coordinates of $\mathbb{T}^m$.

This behaviour can be guessed by doing a Fourier expansion in the compact directions of a solution to $\Delta u = f$: the zero mode satisfy the equation $\Delta_x u_0 = f_0$, and the non zero mode satisfy an equation of the form $\Delta_x u_k + m_k u_k = f_k$, with $m_k > 0$, depending on the mode. Therefore, we see that for the non zero modes, the standard weighted Sobolev spaces in which the more derivative you take, the more decay you get, are not adapted. In [41] we introduce the following definition.

Definition 3.1.1. *The weighted Sobolev space $W^p_{s,\delta,\gamma}(\mathbb{R}^n\times\mathbb{T}^m)$ is the completion of the space C_0^∞ of compactly supported smooth functions for the norm*

$$\|u\|^p_{W^p_{s,\delta,\gamma}(\mathbb{R}^n\times\mathbb{T}^m)} = \sum_{0\leq|\beta|\leq s}\int_{\mathbb{R}^n}|\partial_x^\beta \bar{u}|^p\langle x\rangle^{p(\delta+|\beta|)}\,dx + \sum_{0\leq|\beta|\leq s}\int_{\mathbb{R}^n\times\mathbb{T}^m}|\partial^\beta_{x,\theta}(u-\bar{u})|^p\langle x\rangle^{p\gamma}\,dxd\theta,$$

where $\bar{u}(x) = \frac{1}{vol(\mathbb{T}^m)}\int_{\mathbb{T}^m}u(x,\theta)\,d\theta$ is the average on $\mathbb{T}^m$ for a fixed x and $\langle x\rangle = \sqrt{1+|x|^2}$. Here, $1\leq p<\infty$ and $\delta,\gamma\in\mathbb{R}$.

These weighted Sobolev spaces enjoy product and embedding properties, described in our paper [41].

3.1.2 Main result

Our result on the resolution of the constraint equations on product spaces is the following

Theorem 3.1.2. *Let p be such that $\frac{n+m}{p}<1$. Let g be a metric on $M=\mathbb{R}^n\times\mathbb{T}^m$ such that $g_{ij}-\zeta_{ij}\in W^p_{2,\sigma,\lambda}(M)$, with ζ the flat metric $\zeta = dx^2+d\theta^2$. We assume the following conditions on σ,λ*

$$-\frac{n}{p}<\sigma \quad \text{and} \quad 2-\frac{n+m}{p}<\lambda.$$

Let $\tau\in W^p_{1,\delta+1,\gamma}$, U a vector field in $W^p_{1,\delta+1,\gamma}$ and $R_g\in W^p_{0,\delta+2,\gamma}$ bounded, non negative, with

$$0<\gamma,\quad -\frac{n}{p}<\delta<-\frac{n}{p}+n-2,\quad \frac{n}{p}+\delta+2<2\gamma,$$

together with the coupling conditions

$$\delta+2<\gamma+\lambda,\quad \gamma<\frac{n}{p}+\delta+\lambda.$$

If $||d\tau||_{W^{p}{}_{0,\delta+2,\gamma}}\leq\varepsilon$ with $\varepsilon>0$ sufficiently small, there exists a solution (φ,W) to (3.1.1), with $\varphi=A+u$, $u\in W^p_{2,\delta,\gamma}$, $A>0$, $\varphi>0$ and $W\in W^p_{2,\delta,\gamma}$ a vector field.

Comments on Theorem 3.1.2

- The main novelty of our work is actually the study of the linear operators Δ_g and $\overrightarrow{\Delta}_g$ on the product spaces. The study of the constraint equations in themselves follow then with the methods developed in the asymptotically euclidean case (see for instance [13]). In particular, we believe that our result still holds for a variety of matter sources.
- We wrote our theorem on manifolds diffeomorphic to $\mathbb{R}^n \times \mathbb{T}^n$, with one chart and one end, but it can be generalized to asymptotically flat manifolds with several ends, as in the asymptotically euclidean case.
- It would be interesting to generalize our result to $\mathbb{R}^n \times K$ with K any compact manifold.

In the rest of this section, we will only focus on the inversion of elliptic operators on the product space.

3.1.3 Elliptic theory on the product manifold

In order to construct solutions of the constraint equations, we must first understand the behaviour of the Laplace-Beltrami operator Δ_g and of the conformal Laplacian $\overrightarrow{\Delta}_g$ on the weighted Sobolev spaces we have defined in $M = \mathbb{R}^n \times \mathbb{T}^m$. We begin by looking at a wider class of operators.

Definition 3.1.3. *Let P be a homogeneous self-adjoint second order elliptic operator with constant coefficient, acting on scalar functions or on vector fields of $M = \mathbb{R}^n \times \mathbb{T}^m$. We write*

$$P(u) = \sum_{|\alpha|=2} B^{(\alpha)} \nabla^\alpha u$$

where the $B^{(\alpha)}$ are either scalars or matrices. For a vectorial operator, the ellipticity condition means that for all $\xi \in \mathbb{R}^{n+m}$ with $\xi \neq 0$, $v \mapsto \sum_{|\alpha|=2} \xi^\alpha B^{(\alpha)}(v)$ is an ismorphism on $\mathbb{R}^{n+m}$. Let $s \geq 2$. We say that a second order elliptic operator L is asymptotic to P in $W^p_{s,\sigma,\lambda}$ if

$$L(u) = \sum_{|\alpha|\leq 2} a^{(\alpha)} \nabla^\alpha u$$

with

- *for $|\alpha| = l$, $a^{(\alpha)} - B^{(\alpha)} \in W^p_{s,\sigma,\lambda}$,*
- *for $|\alpha| < l$, $a^{(\alpha)} \in W^p_{s+|l|-2,\sigma+|l|,\lambda}$.*

IIf P_0 is an operator in $\mathbb{R}^n$, we define similarly the notion of being asymptotic to P_0 in $W^p_{s,\delta}$.

These operators verify a series of properties described below.

Theorem 3.1.4. *Let P be a second order homogeneous elliptic self-adjoint operator, acting on functions or vector fields of $\mathbb{R}^n \times \mathbb{T}^m$ with constant coefficients, and L be a second order elliptic operator asymptotic to P in $W^p_{2,\sigma,\gamma}$ with $\frac{n+m}{p} < 2$, $\sigma > -\frac{n}{p}$ and $\lambda > 0$. Let δ, γ be such that*

$$\delta + 2 < \gamma + \lambda, \quad \gamma < \frac{n}{p} + \delta + \lambda.$$

Then:

- *$L : W^p_{2,\delta,\gamma} \to W^p_{0,\delta+2,\gamma}$ is a continuous map,*
- *If $-\delta - \frac{n}{p} \notin \mathbb{N}$ then $L : W^p_{2,\delta,\gamma} \to W^p_{0,\delta+2,\gamma}$ has finite dimensional kernel and closed range, and there exists C and R such that for all $u \in W^p_{2,\delta,\gamma}$*

$$\|u\|_{W^p_{2,\delta,\gamma}} \leq C \left(\|Lu\|_{W^p_{0,\delta+2,\gamma}} + \|\overline{u}\|_{L^p(B_R)} \right). \tag{3.1.2}$$

In the particular case of Δ_g or $\overrightarrow{\Delta}_g$, we prove that with sufficient decay, the operators are isomorphisms.

Theorem 3.1.5. *Let $g \in W^p_{2,\sigma,\lambda}$. In addition to the hypothesis of Theorem 3.1.4, we assume*

$$\gamma > 0, \quad -\frac{n}{p} < \delta < -\frac{n}{p} + n - 2, \quad \lambda + \frac{n+m}{p} > 2.$$

Let $h \in W^p_{0,\delta+2,\gamma}$ be a non negative bounded function. Then $\Delta_g + h$ and $\overrightarrow{\Delta_g}$ are isomorphisms $W^p_{2,\delta,\gamma}(\mathbb{R}^n \times \mathbb{T}^n) \to W^p_{0,\delta+2,\gamma}(\mathbb{R}^n \times \mathbb{T}^n)$. Moreover, we have

$$\|u\|_{W^p_{2,\delta,\gamma}} \leq C\|Lu\|_{W^p_{0,\delta+2,\gamma}},$$

with L being $\Delta_g + h$ or $\overrightarrow{\Delta_g}$.

Some ideas of the proof of Theorem 3.1.4

The proof of Theorem 3.1.4 is done in two steps. The first one is to prove it for P, a homogeneous self-adjoint elliptic operator with constant coefficients. The second one consists in extending the properties proved for P to elliptic operators which are asymptotic to P : this second step is in fact very similar to what is done in asymptotically euclidean manifold (see [5]).

A key estimate in the first step is given by the following proposition.

Proposition 3.1.6. *Let $\gamma, \delta \in \mathbb{R}$, and $u \in W^p_{2,\delta,\gamma}(\mathbb{R}^n \times \mathbb{T}^m), f \in W^p_{0,\delta+2,\gamma}(\mathbb{R}^n \times \mathbb{T}^m)$ such that $P(u - \overline{u}) = f - \overline{f}$ then*

$$\|u - \overline{u}\|_{W^p_{2,\delta,\gamma}} \lesssim \|f - \overline{f}\|_{W^p_{0,\delta+2,\gamma}}.$$

In particular, $P(u - \overline{u}) = 0$ implies $u - \overline{u} = 0$.

The proof of this proposition relies on harmonic analysis. For the analysis of the zero mode, we use the corresponding results on asymptotically euclidean manifolds (see Appendix 2 of [11] and Theorem 1.10 in [5]).

Some ideas of the proof of Theorem 3.1.5

To prove Theorem 3.1.5, the key point is to prove that the operators Δ_g and $\overrightarrow{\Delta}_g$ are injective on the considered weighted Sobolev spaces. For Δ_g this follows easily by using the maximum principle. On the other side, the Kernel of $\overrightarrow{\Delta}_g$ is given by the conformal Killing fields vanishing at infinity : we prove that this set is empty on $(\mathbb{R}^m \times \mathbb{T}^m, g)$ with g asymptotic to the flat metric with sufficient decay rates, following the proof of [59] in the asymptotically euclidean case.

3.2 Global existence for a system of quasilinear equations on a product space

In [40], we address the problem of global existence of small solutions to a certain class of quasilinear systems of wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$. Here the choice of a one dimensional compact direction is done for sake of simplicity, but our result can be adapted without difficulties to $\mathbb{R}^{1+3} \times \mathbb{T}^d$. It would only require working in Sobolev spaces of higher regularity.

The system under consideration has the following form

$$\begin{cases} \Box_{x,y} u + u\partial_y^2 u = \sum_{1\le i,j\le 2} \mathbf{N}_1^{ij}(w_i, w_j) \\ \Box_{x,y} v + u\partial_y^2 v = \sum_{1\le i,j\le 2} \mathbf{N}_2^{ij}(w_i, w_j) \end{cases} \qquad (t,x,y) \in \mathbb{R}^{1+3} \times \mathbb{S}^1 \tag{3.2.1}$$

with initial conditions set at time $t_0 = 2$

$$(u,v)(2,x,y) = (\phi_0, \psi_0)(x,y), \qquad (\partial_t u, \partial_t v)(2,x,y) = (\phi_1, \psi_1)(x,y). \tag{3.2.2}$$

We remark here that our choice to set the initial data at time $t_0 = 2$ over the conventional $t_0 = 0$ is more convenient for our computations and comes at no expense as the system (3.2.1) is invariant under time translations.

In the system (3.2.1), $\Box_{x,y} = -\partial_t^2 + \Delta_x + \partial_y^2$ denotes the D'Alembertian operator in the (t,x,y) variables where $t \in \mathbb{R}$ is the time coordinate, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ are the Cartesian coordinates and $y \in \mathbb{S}^1$ is the periodic coordinate. The nonlinearities $\mathbf{N}_1^{ij}(\cdot,\cdot), \mathbf{N}_2^{ij}(\cdot,\cdot)$ are linear combination of the following quadratic null forms:

$$\begin{aligned} Q_0(\phi,\psi) &= \partial_t\phi\,\partial_t\psi - \nabla_x\phi\cdot\nabla_x\psi \\ Q_{ij}(\phi,\psi) &= \partial_i\phi\,\partial_j\psi - \partial_i\phi\,\partial_i\psi \qquad 1 \le i < j \le 3 \\ Q_{0i}(\phi,\psi) &= \partial_t\phi\,\partial_i\psi - \partial_i\phi\,\partial_t\psi \qquad 1 \le i \le 3 \end{aligned} \tag{3.2.3}$$

where ∂_j denotes the derivative ∂_{x_j} in the jth direction for $j = \overline{1,3}$, and $W = (w_1, w_2)(u,v)$. With this notation, we will write the system (3.2.1) under the form

$$\Box_{x,y} W + u\partial_y^2 W = N(W,W).$$

The main result we present in this paper asserts the global existence of solutions to (3.2.1) when the initial data are small and localized real functions. Our result also extends to semilinear interactions of the form $\partial_\alpha w_i \cdot \partial_y w_j$ with ∂_α being any of the derivatives in the (t,x,y)-variables.

We see this problem as a toy model for Einstein equations in the following way. We recall that Einstein equations in wave coordinates can be written

$$\Box_g g_{\mu\nu} = P_{\mu\nu}(g)(\partial g, \partial g), \tag{3.2.4}$$

where $P_{\mu\nu}(g)(\partial g, \partial g)$ are quadratic forms. These forms do not have the null structure, but the weak null structure (see [57]). Also, the structure of the quasilinear terms can be seen by using in addition the wave coordinate condition. Thanks to these observations, the stability of Minkowski in $\mathbb{R}^{1+3}$ in wave coordinate has been proved by Lindblad and Rodinianski in [57], and by Le Floch and Ma in the presence of a Klein-Gordon field in [55]. Our toy model consists in keeping in (3.2.4) only the semi linear terms with the null structure, and the quasilinear term $g^{yy}\partial_y^2 g_{\mu\nu}$, where y

is the coordinate on $\mathbb{S}^1$, which can be studied without using additional structure from the wave coordinate condition.

As noted in the introduction, after a Fourier expansion of our solution W in the periodic variable y, we obtain an infinite system of coupled Klein-Gordon equation, with a wave equation for the zero mode. Therefore, our analysis is inspired from many works

- on the wave equation: Alinhac [1], Klainerman [49],
- on Klein-Gordon equation: Klainerman [48],
- on wave-Klein Gordon systems: Dong-Wyatt [20], Klainerman-Wang-Yang [53], Ionescu-Pausader [42].

Let us mention also that wave equations on product spaces also appear in other contexts, for instance when studying the propagation of waves along infinite homogeneous waveguides. In particular, other global existence results for small data on product spaces have been done in [61, 62, 22].

3.2.1 The main result

In order to describe the initial data considered for our problem we introduce the energy space $\mathcal{H}^0$ endowed with the norm

$$\|(u[t], v[t])\|^2_{\mathcal{H}^0} := \|u\|^2_{H^1_{xy}} + \|u_t\|^2_{L^2_{xy}} + \|v\|^2_{H^1_{xy}} + \|v_t\|^2_{L^2_{xy}}$$

and the higher order energy spaces $\mathcal{H}^n$ for $n \geq 1$ endowed with the norm

$$\|(u[t], v[t])\|^2_{\mathcal{H}^n} := \sum_{|\alpha| \leq n} \left\|(\partial^\alpha_{xy} u[t], \partial^\alpha_{xy} v[t])\right\|^2_{\mathcal{H}^0}$$

where we use the following notation for the Cauchy data in (3.2.2) at time t:

$$(u[t], v[t]) := (u(t), u_t(t), v(t), v_t(t)).$$

The global well-posedness result that is the object of this paper is proved under some decay assumptions on the initial data. A premilinary version of our main theorem states the following

Theorem 3.2.1. *Assume the initial data* $(u[2], v[2])$ *for* (3.2.1) *satisfy*

$$\sum_{k=0}^{5} \left\| \langle x \rangle^{k+\frac{\alpha+1}{2}} \partial^k_{xy} (u[2], v[2]) \right\|_{\mathcal{H}^0} \leq \varepsilon \ll 1,$$

for some positive fixed α *and* $\langle x \rangle = \sqrt{1+|x|^2}$. *Then the system* (3.2.1) *is globally well-posed in the space* $\mathcal{H}^5$.

3.2.2 Overview of the proof

The proof of Theorem 3.2.1 relies on the vector field method. First, we look for the vector fields commuting with the wave operator $\Box_{t,x,y}$ on $\mathbb{R}^{3+1} \times \mathbb{S}^1$: they are

- the translations $\partial_t, \partial_x, \partial_y$,
- the rotations and the boosts of $\mathbb{R}^{3+1}$: $\Omega_{ij} = x_i\partial_j - x_j\partial_i$, $\Omega_{0i} = t\partial_i + x_i\partial_t$.

We denote by Z these commuting vector fields. Unlike in $\mathbb{R}^{1+3}$, the wave operator does not commute with the scaling vector field $S = t\partial_t + r\partial_r$. In fact, our system admit the same collection of commuting vector field than a Klein-Gordon equation.

The general strategy to show global well-posedness with the vector field method is the following. First we assume that up to a time $T > 0$, we have $\|\partial Z^I W\|_{L^2_{x,y}} \leq 2C\varepsilon$, for a determined number of vector fields Z, belonging to the set of commuting vector fields: this is called the bootstrap assumption. Then thanks to a Klainerman-Sobolev inequality, we recover pointwise decay for the solution. It remains to improve the bootstrap assumptions on $[0, T]$ by performing an energy estimate, which concludes the proof of global existence.

This method has of course to be adapted. First, we are not able to propagate uniform estimates in $\mathcal{H}^5$ and so we have to allow a small growth in t^ρ with $\rho \ll 1$. More importantly, applying Klainerman-Sobolev estimate is an issue here, since the standard version involves the scaling vector field. This problem has been overcome already for Klein-Gordon equations, for compactly supported data, by using a Klainerman-Sobolev on hyperboloids, whose proof can be found in [55].

Lemma 3.2.2. *Let $f = f(t, x)$ be a sufficiently regular function in the interior region $t > |x|$. For all (t, x) in this region, let $s = \sqrt{t^2 - |x|^2}$ and $B(x, t/3)$ be the ball centered at x with radius $t/3$. Then*

$$|f(t,x)|^2 \leq Ct^{-3} \sum_{|\gamma|\leq 2} \int_{B(x,t/3)} \left| Z^\gamma f(\sqrt{s^2 + |y|^2}, y)\right|^2 dy$$

where C is a positive universal constant and the vector fields Z are taken in the set of Ω_{0i} for $i = 1, 2, 3$.

In the region $t < |x|$ we have to use an other approach, based on weighted estimates. We will describe briefly the strategy in the exterior in Section 3.2.3., and in the interior in the Section 3.2.4.

3.2.3 Strategy in the exterior

We foliate the exterior region $\mathcal{D}^{\text{ex}} = \{(t, x, y),\ |x| \geq t - 1\}$ by constant time slices Σ_t^{ex}

$$\Sigma_t^{\text{ex}} := \{x \in \mathbb{R}^3 : |x| \geq t - 1\} \times \mathbb{S}^1,$$

and use weighted energy estimates, described in the following lemma, which can be proved simply by multiplying the wave equation by $(2 + r - t)^{1+\alpha}\partial_t W$, with $r = |x|$, and integrating over $\mathcal{D}^{\text{ex}}$. We use the notation

$$|\partial W|^2 = |\partial_t W|^2 + |\nabla W|^2 + |\partial_y W|^2, \quad |\mathcal{T} W|^2 = \sum_{i=1}^{3} \left|\partial_i W + \frac{x_i}{r}\partial_t W\right|.$$

Lemma 3.2.3. *We have the following estimate, for a solution to the equation* $\Box_{t,x,y}W = F$,

$$\begin{aligned}&\int_{\Sigma_t^{ex}}(2+r-t)^{1+\alpha}|\partial W|^2dxdy + \int_2^t\int_{\Sigma_s^{ex}}(2+r-s)^{\alpha}(|\mathcal{T}W|^2+|\partial_y W|^2)dxdyds\\ &\qquad + \int_{\mathcal{C}_{[2,t]}}(2+r-t)^{\alpha+1}\left(|\mathcal{T}W|^2+|\partial_y W|^2\right)d\sigma dyds\\ &\lesssim \int_{\Sigma_2^{ex}} r^{1+\alpha}|\partial W|^2dxdy + \int_2^t\int_{\Sigma_s^{ex}}(2+r-t)^{1+\alpha}|F\partial_t W|dxdyds\end{aligned}$$

where $C_{[2,t]}$ *is the null boundary of* $\mathcal{D}^{ex}$ *and* $d\sigma$ *is the area element of the sphere* $\mathbb{S}^2$.

We see with this lemma that the use of weighted energy yields additional integrated estimates for derivatives tangential to the outgoing null cones, $\mathcal{T}u$ and $\partial_y u$. This is a fundamental ingredient for the global existence in the exterior. The other main ingredients are the following Klainerman-Sobolev type inequalities

Lemma 3.2.4. *Let* $\beta \in \mathbb{R}$. *For any sufficiently smooth function* w *we have*

$$(2+r-t)^{\beta}r^2|w(t,x,y)|^2 \lesssim \int_{\Sigma_t^{ex}}(2+r-t)^{\beta+1}(\partial_r\mathcal{Z}^{\leq 2}w)^2 + (2+r-t)^{\beta-1}(\mathcal{Z}^{\leq 2}w)^2\,dxdy.$$

and also

$$(2+r-t)^{\beta}r^2|w(t,x,y)|^2 \lesssim \int_{\Sigma_t^{ex}}(2+r-t)^{\beta}\Big((\partial_r Z^{\leq 2}w)^2 + (Z^{\leq 2}w)^2\Big)dxdy.$$

where we denote $Z^{\leq 2}$ *for* $\sum_{|\gamma|\leq 2} Z^{\gamma}$.

3.2.4 Strategy in the interior

In the interior, the strategy based only on energy estimates and Klainerman-Sobolev inequality on hyperboloids is not sufficient to close the argument. We need to introduce the following decomposition

$$W = W_0 + \mathtt{W}, \qquad W_0 = \int_{\mathbb{S}^1} W(t,x,y)dy. \tag{3.2.5}$$

We note that W_0 satisfies a wave equation on $\mathbb{R}^{1+3}$

$$\Box_{t,x}W_0 = \int_{\mathbb{S}^1}(N(W,W) - u\partial_y^2 W)dy.$$

3.2.5 Specific estimates for W_0

This decomposition permits to point out one of the main difficulties in the analysis, which is the quasilinear term $u_0\partial_y^2\mathtt{W}$. Indeed, the energy estimate for the wave component only involves derivatives of W_0, and as a consequence, Klainerman-Sobolev inequality only yields a pointwise estimate for ∂u_0. However, since W_0 satisfies a wave equation on $\mathbb{R}^{1+3}$, we can use well-known tools to obtain additional informations on ZW_0 for any commuting vector field Z.

The first one is the use of the conformal energy. This is obtained by multiplying the wave equation by Ku, where K is the Morawetz vector field

$$K = (t^2 + r^2)\partial_t + 2rt\partial_r,$$

and integrating over the domain. More precisely, we have the following estimate on constant time foliation.

Lemma 3.2.5. *We have the following estimate, for a solution to the equation* $\Box_{t,x} W_0 = F_0$

$$E^c(t, W_0)^{\frac{1}{2}} \lesssim E^c(2, W_0)^{\frac{1}{2}} + \int_2^t \|(s + r)F(s, x)\|_{L^2_x} ds,$$

where the conformal energy is defined by

$$E^c(t, W_0) = \int |\mathcal{S}W_0 + 2W_0|^2 + \sum_{i=0}^{3} |\Omega_{0i}W_0|^2 dx.$$

We see with the expression of the conformal energy that we can have directly a control on $\Omega_{0i}W_0$. Moreover, a similar estimate can be obtained on hyperboloids.

Remark 3.2.6. *There is a small complication with the use of the conformal energy, which is that contrary to the standard energy for which we are able to prove uniform boundedness in the exterior, the conformal energy has a logarithmic growth. Because of that, we can not consider an initial value problem posed on the hyperboloid* $\mathcal{H}_1$ *: the conformal energy on it is not defined. We overcome this difficulty by cutting our hyperboloids* $\mathcal{H}_s$ *at time* $t \sim s$*, the remaining portion being handled as the exterior region estimates.*

The second tool we use are the $L^\infty - L^\infty$ estimates, due to Alinhac [1], and which can also be found in [55].

Lemma 3.2.7. *Let* W_0 *be the solution to* $\Box_{tx} W_0 = F_0$ *with zero initial data and suppose that* F_0 *is spatially compactly supported satisfying the following pointwise bound*

$$|F_0(t, x)| \le Ct^{-2-\nu}(t - |x|)^{-1+\mu}$$

for some positive constant C*. Then*

$$|W_0(t, x)| \lesssim \frac{C}{\mu\nu}(t - |x|)^{\mu-\nu} t^{-1}.$$

With this lemma, we see that in order to obtain a sharp $\frac{1}{t}$ bound for W_0, we need a sharp bound for $\mathtt{W}$. This is explained in the next subsection.

3.2.6 Specific estimates for $\mathtt{W}$

Since the components ZW_0 have at most decay in $\frac{1}{t}$, and because of the presence of the term $u_0\partial_y^2\mathtt{W}$ in the equation, the energy for $Z^I W$, with $|I| \ge 1$ may have a logarithmic growth in time. As a consequence, the estimates obtained via the Klainerman-Sobolev inequalities have also a logarithmic growth, which, once injected in Lemma 3.2.7 yield uncontrolled growth. This issue has already been seen, and overcome in [20]. The strategy relies on the use of sharp pointwise estimates for solutions to Klein-Gordon equation with varying mass. We state here a version which is adapted to our problem on $\mathbb{R}^{1+3} \times \mathbb{S}^1$.

Proposition 3.2.8. *Assume that* $\mathbf{W}$ *is a solution of the following equation*

$$\Box_{x,y}\mathbf{W} + u\Delta_y\mathbf{W} = \mathbf{F}, \qquad (t,x,y) \in \mathbb{R}^{1+3}\times\mathbb{S}^1 \tag{3.2.6}$$

such that $\int_{\mathbb{S}^1}\mathbf{W}dy = 0$. *For every fixed* (t,x) *in the cone* $\mathcal{C} = \{t > r\}$, *let* $s = \sqrt{t^2 - r^2}$ *and* Y_{tx}, A_{tx}, B_{tx} *be the functions defined as follows*

$$Y_{tx}^2(\lambda) := \int_{\mathbb{S}^1} \lambda \left|\frac{3}{2}\mathbf{W}_\lambda + (\mathcal{S}\mathbf{W})_\lambda\right|^2 + \lambda^3(1+u_\lambda)|\partial_y\mathbf{W}_\lambda|^2\,dy, \tag{3.2.7}$$

$$A_{tx}(\lambda) := \sup_{\mathbb{S}^1}\left|\frac{1}{2\lambda}(\mathcal{S}u)_\lambda\right| + \sup_{\mathbb{S}^1}|\partial_y u_\lambda|, \tag{3.2.8}$$

$$B_{tx}^2(\lambda) = \int_{\mathbb{S}^1}\lambda^{-1}\left|(R\mathbf{W})_\lambda\right|^2\,dy, \tag{3.2.9}$$

where $f_\lambda(t,x,y) = f\left(\frac{\lambda t}{s}, \frac{\lambda r}{s}, y\right)$ *and*

$$R\mathbf{W}(t,x,y) = s^2\bar{\partial}^i\bar{\partial}_i\mathbf{W} + x^ix^j\bar{\partial}_i\bar{\partial}_j\mathbf{W} + \frac{3}{4}\mathbf{W} + 3x^i\bar{\partial}_i\mathbf{W} - s^2\mathbf{F}.$$

Then $\mathbf{W}$ *satisfies the following inequality in the interior region*

$$s^{\frac{3}{2}}\left(\|\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\mathbf{W}\|_{L^2(\mathbb{S}^1)}\right) + s^{\frac{1}{2}}\|\mathcal{S}\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim \left(Y_{tx}(2) + \int_2^s B_{tx}(\lambda)d\lambda\right)e^{\int_2^s A_{tx}(\lambda)\,d\lambda}.$$

3.3 Perspectives

The next step in the study of Einstein equation with compact directions is the nonlinear stability of the solution $(\mathbb{R}^{3+1}\times\mathbb{T}^d, g)$ with $g = m + dz^2$, where m is the Minkowski metric on $\mathbb{R}^{3+1}$ and dz^2 is the flat metric on the torus. For this, the work of LeFloch and Ma [55] on Einstein-Klein Gordon equations could be a good reference on how to deal with the weak null structure.

Another interesting question could be the study of a similar problem near Schwarzschild solution. Then one should face two major difficulties: the fact that Klein Gordon equation on a Schwarzschild black-hole is not fully understood, and the Gregory-Laflamme instability of the Schwarzschild black string (see for instance [17]), which is a solution to Einstein vacuum equation given by

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\sigma^2 + dz^2.$$

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