Wave equations and general relativity

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Chapter 1

Introduction

General relativity is a theory of gravitation, introduced by Einstein in 1915, which relies on deep mathematical concepts. To have a first flavour, we will make a parallel with electromagnetism.

1.1 Electromagnetism

Before the ninetieth century, the electrical and magnetic forces where thought to be independent phenomena. The Coulomb law of electrostatics says that the force F_{AB} applied by a charge q_A, situated at a point A, on a charge q_B, situated at a point B, is given by

$$F_{AB} = \frac{q_A q_B}{4\pi\varepsilon_0 r^2} u,$$

where $r = |AB|$ and $u = \frac{\overrightarrow{AB}}{r}$. Poisson expressed later this law in term of an electric potential V, created by a density of charge ρ. The electric force which acts on the point charge (B, q_B) is then $-q_B \nabla V$ where V satisfies the Poisson equation

$$\Delta V = -\frac{\rho}{\varepsilon_0}.$$

This equation was then completed by three others, to obtain the full set of Maxwell equations which describe the classical theory of electromagnetism

$$\begin{aligned}
\nabla \wedge E &= -\frac{\partial B}{\partial t} && \text{Faraday's law,}\\
\nabla \wedge B &= \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} && \text{Ampère-Maxwell's law,}\\
\nabla . E &= \frac{\rho}{\varepsilon_0} && \text{Coulomb's law,}\\
\nabla . B &= 0 && \text{Gauss's law.}
\end{aligned}$$

From these equations, one can show that E and B satisfy **wave equations**

$$\begin{aligned}
\partial_t^2 E - c^2 \Delta E &= 0,\\
\partial_t^2 B - c^2 \Delta B &= 0.
\end{aligned}$$

The electromagnetic waves propagate with a speed c, which is a constant independent of the choice of inertial frame. This fact was a contradiction with Newtonian mechanics, and led to the theory of special relativity, formulated by Einstein in 1905. In this theory, space and time are described by Minkowski space-time : this is $\mathbb{R}^4$ equipped with a quadratic form

$$m = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

In Minkowski space-time, which is a particular example of **Lorentzian manifold**, the causal future of a point, which is the set of points which can be reached without going faster than the speed of light, is a cone.

1.2 Gravitation

Before general relativity, gravitation was modelled by Newton's law. A point mass (A, m_A) acts on a point mass (B, m_B) through the force

$$F_{AB} = -\frac{G m_A m_B}{r^2} u.$$

This can be expressed in term of the gravitational potential ϕ. The force applied by a density of mass ρ on a point mass (B, m_B) is $-m_B \nabla\phi$, where ϕ satisfy the Poisson equation

$$\Delta\phi = 4\pi G\rho.$$

In Newton's universal law of gravitation, the mass m_A acts at distance on the mass m_B, which is an apparent contradiction with special relativity. One can note that in electrostatics there is also a principle of action at a distance. However when Coulomb law is completed with the whole set of Maxwell equations, we see that the propagation of the electromagnetic field obeys wave equations with speed c. For gravitation, the resolution of this paradox has been done by Einstein in 1915 through a revolutionary change of point of view : the theory of general relativity. In this theory, the gravitation is not a force but is encoded in the geometry of the space-time, which is described by a Lorentzian manifold (M, g) : the bodies subject only to the gravitation follow the **geodesics** in this new geometry. The Lorentzian metric g must obey Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

In these equations, $R_{\mu\nu}$ describe the **curvature** of the space-time, and $T_{\mu\nu}$ is the stress energy tensor : its form depends on the fields which are considered. Newton's equation is not part of Einstein equations, but it can be obtained by taking well chosen limits.

Einstein equations are invariant by change of coordinates. In a particular set of coordinates, they can be written as wave equations on the metric coefficients. In general relativity, the deformations of space-time propagate in the form of gravitational waves with the speed of light.

The aim of this course is

- to introduce the geometrical notions which are fundamental in General Relativity,
- to have a first approach on the physical meaning and implication of the theory,

- to see that studying general solutions to Einstein equations consist in solving evolutionary partial differential equations,
- to introduce the analytical background necessary to solve wave equations, and therefore Einstein equations.

We refer to the book [8] and [6] for more details on Riemannian geometry, and to [7] for Lorentzian geometry. We refer to [9] and [3] for a complete introduction to General Relativity and we refer to [4] and [1] for more details on the solving of partial differential equations. We also refer to the following lectures notes :

- Jacques Smulevici : Lectures on Lorentzian Geometry and hyperbolic pdes,
- Jérémie Szeftel : Introduction à la relativité générale d'un point de vue mathématique,
- Jonathan Luk : Introduction to Nonlinear Wave Equations.

Chapter 2

Riemannian geometry

In this chapter we will introduce the notions of geometry which are needed to formulate Einstein equations and to study them. These notions are useful in many areas of mathematics, so we will introduce a very general setting in this chapter.

2.1 Manifolds

In general relativity the space-time is a differential manifolds. It is a topological space which is locally identified with $\mathbb{R}^n$. Let us be more precise.

2.1.1 Charts and atlas

Definition 2.1.1. *Let M be a topological space. A topological atlas of dimension n is a family (U_i, ϕ_i) such that*

- *U_i are open,*
- $\cup U_i = M$
- *$\phi_i : U_i \to \Omega_i$ where Ω_i is an open set of $\mathbb{R}^n$ is an homeomorphism.*

The map ϕ_i is called chart or coordinate system, the set U_i is the domain of the chart and the maps $\phi_i \circ \phi_j^{-1}$ are called change of charts.

Definition 2.1.2. *A topological manifold M of dimension n is a topological Haussdorff space (séparé) equipped with a countable atlas.*

Proposition 2.1.3. *A topological manifold is locally compact, locally path connected, separable, paracompact, metrizable.*

Remark 2.1.4. *We can define the notion of atlas for M a general set. It is a family of pairs (U_i, ϕ_i) such that*

- *$U_i \subset M$ and $\cup U_i = M$,*
- *$\forall i$ $\phi_i : U_i \to \Omega_i$, with Ω_i an open set of $\mathbb{R}^n$*

- *$\forall i, j : \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is an homeomorphism.*

The data of an abstract atlas on M yields a topology on M : U is open if for all U_j, $\phi_j(U_j \cap U)$ is open in $\mathbb{R}^n$. If this topology is Haussdorff, and if I is countable, we obtain a structure of manifold on M.

An atlas $\{U_a, \phi_a\}$ is called smooth if all chart transitions

$$\phi_b \circ \phi_a^{-1} : \phi_a(U_a \cap U_b) \to \phi_b(U_a \cap U_b)$$

are differentiable of class C^∞. Two atlases are equivalent if their union is still an atlas.

Definition 2.1.5. *A differentiable manifold is a topological manifold which admits a smooth atlas. A smooth structure is an equivalence class of smooth atlases*

We can also define C^k manifolds by requiring that the transition map $\phi_b \circ \phi_a^{-1}$ are C^k where they are defined.

The example of the sphere The sphere $\mathbb{S}^n = \{(x^1, .., x^{n+1}) \in \mathbb{R}^{n+1},\ \sum_{i=1}^{n+1}(x^i)^2 = 1\}$ is a differentiable manifold. One can define charts (U_i, ϕ_i), $i = 1, 2$ as follows. On $U_1 = \mathbb{S}^n \setminus \{0, .., 0, 1\}$ we define

$$\phi_1(x^1, .., x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, .., \frac{x^n}{1 - x^{n+1}}\right),$$

and on $U_2 = \mathbb{S}^{n-1} \setminus \{(0, .., 0, -1)\}$ we define

$$\phi_2(x^1, .., x^{n+1}) = \left(\frac{x^1}{1 + x^{n+1}}, .., \frac{x^n}{1 + x^{n+1}}\right).$$

We can check that $(U_i, \phi_i)_{i=1,2}$ is a smooth atlas on the sphere.

Definition 2.1.6. *Let (M, N) be two differentiable manifolds, of dimension d and d'. A map $h : M \to N$ is C^k if for all charts (U_a, ϕ_a) on M and (V_b, ψ_b) on N, the map $\psi_b \circ h \circ \phi_a^{-1}$ is of class C^k on the open set of $\mathbb{R}^d$ where it is defined.*

Remark 2.1.7. *Since the transition map are smooth, to check whether a map $h : M \to N$ is C^k, it is sufficient to prove that for all $p \in M$ there exists a chart (U_a, ϕ_a) with $p \in U_a$, and a chart (V_b, ψ_b) with $h(p) \in V_b$ such that $\psi_b \circ h \circ \phi_a^{-1}$ is C^k.*

2.1.2 Submanifolds

Definition 2.1.8. *Let M be a d dimensional manifold. A subset X of M is a submanifold of dimension d' if for every point $x \in X$, there exists a chart (U, ϕ) of M with $x \in U$ such that $\phi(X \cap U) = (\mathbb{R}^{d'} \times \{0\}) \cap \phi(U)$.*

A submanifold N inherits a manifold structure : the charts (U_a, ϕ_a) on M induce charts

$$(X \cap U_a, \pi_{\mathbb{R}^{d'}} \circ \phi_a|_{X \cap U_a})$$

on X. We remark that the submanifolds of dimension d of M are given by the open sets of M. More interesting examples are built on submersions and immersions. Let us for the moment consider immersion and submersion in $\mathbb{R}^n$ which will allow us to easily construct examples as submanifolds on $\mathbb{R}^n$.

Definition 2.1.9. *Let U be an open set of $\mathbb{R}^d$, and let $f : U \to \mathbb{R}^n$ be a map of class C^k.*

- *f is an immersion in $x \in U$ if its differential in x is injective.*
- *f is a submersion in $x \in U$ if its differential in x is surjective.*

Theorem 2.1.10. *Let U be an open set of $\mathbb{R}^d$, and let $f : U \to \mathbb{R}^n$ be a map of class C^k.*

- *If f is an immersion on all points of U then for all $x \in U$ there exists a neighbourhood U_x of x, a neighbourhood $V_{f(x)}$ of $f(x)$ and a diffeomorphism $\psi : V_{f(x)} \to W$ of class C^k, with W an open set of $\mathbb{R}^n$ such that for all $x \in U_x$*
$$\psi \circ f(x^1, .., x^d) = (x^1, .., x^d, 0, .., 0).$$
- *If f is a submersion on all points of U then for all $x \in U$ there exists a neighbourhood U_x of x, a neighbourhood $V_{f(x)}$ of $f(x)$ and a diffeomorphism $\phi : V \to U_x$ of class C^k, with V an open set of $\mathbb{R}^d$ such that for all $x \in V$*
$$f \circ \phi(x^1, .., x^d) = (x^1, .., x^n).$$

From this theorem, we can obtain the following proposition

Proposition 2.1.11. *Let U be an open set of $\mathbb{R}^d$, and let $f : U \to \mathbb{R}^n$ be a map of class C^k.*

- *If f is injective, proper, and an immersion on all points of U then $f(U)$ is a submanifold of $\mathbb{R}^n$ of dimension d.*
- *Let $y \in \mathbb{R}^n$. If $f^{-1}(y) \neq \emptyset$ and for all $x \in f^{-1}(Y)$ f is a submersion on x then $f^{-1}(y)$ is a submanifold of $\mathbb{R}^d$ of dimension $d - n$.*

Proof. We give the proof of the second point. Let $x \in f^{-1}(Y)$. There exists a neighbourhood U_x of x, a neighbourhood V_y of y and a diffeomorphism $\phi : V \to U_x$ of class C^k, with V an open set of $\mathbb{R}^d$ such that for all $x \in V$
$$f \circ \phi(x^1, .., x^d) = (x^1, .., x^n).$$
Consequently, in $f^{-1}(y) \cap U_x$ we have $(x^1, .., x^n) = y$ and the diffeomorphism $\phi^{-1} : U_x \to V$ is such that
$$\phi^{-1}(f^{-1}(Y) \cap U_x) = (\{y\} \times \mathbb{R}^{d-n}) \cap V.$$
□

Example of submersion We consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x^1, .., x^n) = (x^1)^2 + .. + (x^n)^2$. For $r > 0$, f is a submersion for all $x \in f^{-1}(r)$. Indeed, the differential df_x is the map
$$df_x : h \in \mathbb{R}^n \mapsto \sum_i 2x^i h^i \in \mathbb{R}.$$

This map is surjective when the x^i are not all equal to zero, that is to say for $f(x) \neq 0$. Thanks to Proposition 2.1.11, $f^{-1}(r)$ is then a submanifold of $\mathbb{R}^n$, of dimension $n - 1$. This gives an other proof of the fact that $\mathbb{S}^{n-1}$ is a manifold.

Example of immersion We consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x, y, x^2 + y^2)$. The differential is

$$df_{(x,y)} : h = (h^1, h^2) \in \mathbb{R}^2 \mapsto (h^1, h^2, 2xh^1 + 2yh^2) \in \mathbb{R}^3.$$

Therefore, we see that f is injective, proper, and an immersion on $\mathbb{R}^2$. Consequently, the parabola $f(\mathbb{R}^2)$ is a submanifold of $\mathbb{R}^3$, of dimension 2.

Remark 2.1.12. *We can speak about immersions and submersions between manifolds, by asking that in some charts, the application is an immersion or a submersion, and extend Proposition 2.1.11 to manifolds.*

2.1.3 Tangent vectors

Tangent vectors are easy to define for a submanifold M of $\mathbb{R}^n$: we say that v is tangent to M in x if there exists a curve $c :]-\varepsilon, \varepsilon[\to M$ such that $c(0) = x$ and $\dot{c}(0) = v$. We can define tangent vectors in the more abstract setting following this idea.

Definition 2.1.13. *Let M be a differential manifold and let $x \in M$. Two C^1 paths $c_1 :]-\varepsilon, \varepsilon[\to M$ and $c_2 :]-\varepsilon, \varepsilon[\to M$ such that $c_1(0) = c_2(0) = x$ are equivalent if there exists a chart ϕ in a neighbourhood of x such that $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$. A tangent vector in x is an equivalence class of path for this relation. The set of tangent vectors in x is noted T_xM : it is the tangent space to M in x.*

Let $f : M \to N$ be a smooth map between manifolds. If c is a path through $x \in M$, $f \circ c$ is a path through $f(x) \in N$. Moreover, if c_1 and c_2 are two equivalent paths, then $f \circ c_1$ and $f \circ c_2$ are also two equivalent paths. This allow to define the differential of f in x

$$\begin{aligned} df_x : T_xM &\to T_{f(x)}N \\ [c] &\mapsto [f \circ c] \end{aligned}$$

If (U, ϕ) is a chart in x, The map $d\phi_x$ allows to identify T_xM to $\mathbb{R}^n$.

We can define the **tangent bundle** by $TM = \{(x, X), x \in M, X \in T_xM\}$. We note π the projection $\pi : TM \to M, (x, X) \mapsto x$.

Proposition 2.1.14. *TM is a manifold of dimension $2n$*

Proof. Let (U_i, ϕ_i) be a differential atlas on M. We consider the map

$$\begin{aligned} d\phi_i : \pi^{-1}(U_i) &\to \mathbb{R}^{2n}, \\ (x, X) &\mapsto (\phi_i(x), (d\phi_i)_x(X)) \end{aligned}$$

and check that $(\pi^{-1}(U_i), d\phi_i)$ define a differential atlas on TM, and that the topology induced on TM by this atlas is Haussdorff. $\square$

2.1.4 Vector bundles

TM has a very specific atlas : the charts identify $\pi^{-1}(U)$ to a product $U \times \mathbb{R}^n$, and on each fiber $T_xM = \pi^{-1}(x)$ we have a vectorial space structure. This enters in a more general setting

Definition 2.1.15. *A vector bundle of rank p is a triplet (E, π, B) where E and B are differential manifolds, $\pi : E \to B$ is smooth and each fiber $E_x = \pi^{-1}(x)$ is a vector space with the following condition : for all $x \in B$ there exists an open neighbourhood U of x and a diffeo $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^p$ such that $p_U \circ \psi = \pi$ and $p_{\mathbb{R}^p} \circ \psi|_{E_y} : E_y \to \mathbb{R}^p$ is an isomorphism.*

E is called the total space, B is called the base space and ψ is called a local trivialization. If ψ_1 and ψ_2 are two trivializations then

$$\begin{aligned} \psi_1 \circ \psi_2^{-1} : U_1 \cap U_2 \times \mathbb{R}^p &\to U_1 \cap U_2 \times \mathbb{R}^p \\ (x, v) &\mapsto (x, u_{12}(v)) \end{aligned}$$

where $u_{12} \in GL(\mathbb{R}^p)$.

Example : We can construct a local trivialization of TM in the following way : if $(U, \phi = (x^1, .., x^n))$ is a chart, we can consider, for all $x \in U$,

$$e_i = [t \mapsto \phi^{-1}(\phi(x) + (0, .., t, 0, ..))]$$

For all x, the e_i are independent tangent vectors of T_xM. The local trivialisation of TU we have constructed is

$$\begin{aligned} \psi : \qquad & \pi^{-1}(U) \to U \times \mathbb{R}^n \\ (x, X) \text{ with } X \in T_xM & \mapsto (x, X^1, .., X^n), \text{ with } X = \sum_i X^i e_i. \end{aligned}$$

In this course, we will restrict to the bundles constructed from TM, called tensor bundles.

Definition 2.1.16. *The cotangent bundle is $T^*M = \{(x, \sigma), x \in M, \sigma \in (T_xM)^*\}$.*

We can also define

$$\begin{aligned} &\mathcal{L}(TM, TM) = \{(x, A), x \in M, A \in \mathcal{L}(T_xM, T_xM)\}, \\ &Bil(TM) = \{(x, u), x \in M, u \text{ is a bilinear form on } T_xM\} = \mathcal{L}(TM, T^*M), \\ &\mathcal{L}(\mathcal{L}(TM, TM), TM)... \end{aligned}$$

A handful way of defining these objects is through the tensor product that we recall here. Let U and V be vector spaces. $U \otimes V$ is the vector space generated by the symbols $u \times v$, where $u \in U$ and $v \in V$, quotiented by the subspace generated by the

$$u \otimes (\alpha v + \beta \tilde{v}) - \alpha u \otimes v - \beta u \otimes \tilde{v},$$

and

$$(\alpha u + \beta \tilde{u}) \otimes v - \alpha u \otimes v - \beta \tilde{u} \otimes v.$$

If U is of dimension n, with basis $(e_1, .., e_n)$, and V is of dimension m, with basis $(f_1, .., f_m)$, then $U \otimes V$ is of dimension mn, with basis $e_i \otimes f_j$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Proposition 2.1.17. *The tensor product satisfy the following properties*

- *If $B : U \times V \to W$ is a bilinear map, then their exists a unique linear map $\tilde{B} : U \otimes V \to W$ such that $B = \tilde{B} \circ h$ where $h : U \times V \to U \otimes V$ is defined by*

$$h(u, v) = u \otimes v,$$

- $(U \otimes V) \otimes W = U \otimes (V \otimes W)$. *Therefore we note simply* $U \otimes V \otimes W$.
- $U \otimes V \sim V \otimes U$,
- $\mathcal{L}(U, V) \sim U^* \otimes V$,
- $(U \otimes V)^* \sim U^* \otimes V^*$.

The proof of this proposition is left to the reader but we precise for instance what we mean by $\mathcal{L}(U, V) \sim U^* \otimes V$: their exists a natural isomorphism between the two space. In this case this is the linear map $\phi : U^* \otimes V \to \mathcal{L}(U, V)$ defined by $\phi(u^* \otimes v) = (f \in U \mapsto u^*(f)v \in V)$.

Definition 2.1.18. *A section of a vector bundle (E, π, B) is a smooth map $\sigma : B \to E$ such that $\pi \circ \sigma = id$. The space of sections is denoted $\Gamma(E)$. Locally, a section is a smooth map $U \to \mathbb{R}^p$.*

For instance, if $E = TM$, the sections of E are called vector fields. The sections of T^*M are called 1-forms.

If $f \in C^\infty(M, \mathbb{R})$ we can define the differential of f at x : $df_x \in (T_xM)^*$. The differential of f, $df : M \to T^*M,\ x \mapsto T^*_xM$ is a section of T^*M.

If we have a chart (U, ϕ), the components x^i of ϕ in $\mathbb{R}^n$ are coordinates. We can see them as functions $x^i : U \to M$. The dx^i are locally sections of T^*M which are linearly independent in all point $x \in T^*U$. The dx^i yield a local trivialization of T^*M.

Now that we have the local trivialization of TM, given by the e_i, and the local trivialization of T^*M, a local trivialization of the vector bundles of tensors which are r times contravariant and s times covariant

$$\underbrace{TM \otimes ... \otimes TM}_{\text{r times}} \otimes \underbrace{T^*M \otimes .. \otimes T^*M}_{\text{s times}}$$

by

$$e_{i_1} \otimes ... \otimes e_{i_r} \otimes (dx^{j_1}) \otimes ... \otimes (dx^j_s)$$

A section of this bundle can be written in this local trivialisation

$$T = T^{i_1..i_r}_{j_1..j_s} e_{i_1} \otimes ... \otimes e_{i_r} \otimes (dx^{j_1}) \otimes ... \otimes (dx^j_s),$$

where we use the Einstein summation convention : here a sum is taken for the repeated indices (one up and one down) : $1 \leq i_1, ..i_r, j_1, ..j_s \leq n$. Most of the time the section T will be only denoted by $T^{i_1..i_r}_{j_1..j_s}$.

2.1.5 Vector fields and derivations

In this section, we will see that the vector fields we have defined coincide with an other notion : the derivation. In fact we could also have defined the vector field in term of derivation.

Definition 2.1.19. *Let X be a vector field and $f : M \to \mathbb{R}$ be a smooth function. The Lie derivative of f along X, $\mathcal{L}_X f = df(X)$ is the function $M \to \mathbb{R}$ defined by $(\mathcal{L}_X f)(x) = df_x(X_x)$.*

Definition 2.1.20. *A derivation of $C^\infty(M)$ is a $\mathbb{R}$ linear endomorphism D such that $\forall f, g \in C^\infty(M)$ we have $D(fg) = fDg + gDf$*

If $(U, \phi = (x^1, .., x^n))$ is a chart, we can define a particular derivation on $C^\infty(U)$ by

$$\frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}(f \circ \phi^{-1}(x^1, .., x^n))$$

If we write this expression at a point x we have

$$\begin{aligned}\frac{\partial f}{\partial x^i}(x) &= \frac{d}{dt}(f \circ \phi^{-1}(\phi(x) + (0, .., t, 0, ..))) \\ &= [f \circ c] \text{ where } c : t \mapsto \phi^{1}(\phi(x) + (0, .., x_i, 0, ..)) \\ &= df_x[c] = df(e_i)(x).\end{aligned}$$

Consequently $\frac{\partial f}{\partial x^i} = \mathcal{L}_{e_i} f$. If a vector field can be written locally $X = X^i e_i$ then

$$\mathcal{L}_X f = X^i \frac{\partial}{\partial x_i} f,$$

and we can easily check that $\mathcal{L}_X$ is a derivation. From now on, the local trivialization of TM associated to the coordinates x^i will be noted $\frac{\partial}{\partial x^i}$ instead of e_i. Note that in $\frac{\partial}{\partial x^i}$, the index is down.

2.1.6 Lie derivative

Let X be a vector field. For all x we can define a $c_x : I \to M$, such that $\dot{c_x}(t) = X_{c_x(t)}$ and $c_x(0) = x$. From this we can define the flow ϕ_t of a vector field X by $\phi_t(x) = c_x(t)$. This flow may not be defined for all x and all t, but at least, in each compact set K of M, for t small enough, the flow $\phi_t : K \to M$ exists. By definition, we have $X_x = [c_x(t)]$ and consequently

$$\mathcal{L}_X f = \frac{d}{dt}(f \circ \phi_t(x))|_{t=0}.$$

This give us a way to derive more general tensors. For this we will define the pull back of a vector field.

Definition 2.1.21. *Let $\phi : U \to V$ be a diffeomorphism between two open set of M.*

- *The pull back of a one form $\omega \in T_{\phi(p)}V$ is $(\phi^*\omega)_p = \omega_{\phi(p)} \circ d\phi$.*
- *The pull back of a vector field $X \in T_{\phi(p)}V$ is $(\phi^*X)_p = d\phi^{-1} X_{\phi(p)}$.*

This allows to define the Lie derivative of a vector field.

Definition 2.1.22. *The Lie derivative of a vector field Y along a vector field X is defined by*

$$\mathcal{L}_X Y = (\frac{d}{dt}\phi_t^* Y)|_{t=0}.$$

There exists a more algebraic definition of $\mathcal{L}_X Y$: the Lie bracket .

Definition 2.1.23. *Let X, Y be two vector fields. The Lie bracket $[X, Y]$ is the operation corresponding to the commutator $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$*

In a chart we have

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

This prove that $[X, Y]$ is a vector field.

Proposition 2.1.24. *We have $\mathcal{L}_X Y = [X, Y]$*

Proof. Let ϕ_t be the flow associated to X and ψ_u the flow associated to Y. Using the chain rule for derivation of composed map, we compute

$$\frac{d}{du} \left(\phi_{-t} \circ \psi_u \circ \phi_t(x) \right) = (d\phi_{-t})_{\phi_{-t} \circ \psi_u \circ \phi_t(x)} \left(\frac{d}{du} (\psi_u(\phi_t(x))) \right) = (d\phi_{-t})_{\phi_{-t} \circ \psi_u \circ \phi_t(x)} \left(Y_{\psi_u \circ \phi_t(x)} \right).$$

Evaluating in $u = 0$ we obtain

$$\frac{d}{du} \left(\phi_{-t} \circ \psi_u \circ \phi_t(x) \right) \big|_{u=0} = \phi_t^* Y$$

and therefore

$$\frac{d}{dt} \left(\frac{d}{du} \left(\phi_{-t} \circ \psi_u \circ \phi_t(x) \right) \big|_{u=0} \right) \big|_{t=0} = \mathcal{L}_X Y.$$

Consequently, applying this vector field to a function f we obtain

$$\begin{aligned}
\left(\mathcal{L}_X Y \right)(f) &= df_x(\mathcal{L}_X Y) \\
&= df_x \left(\frac{d}{dt} \frac{d}{du} \left(\phi_{-t} \circ \psi_u \circ \phi_t(x) \right) \right) \\
&= \frac{d}{dt} df_x \left(\frac{d}{du} \left(\phi_{-t} \circ \psi_u \circ \phi_t(x) \right) \right) \\
&= \frac{d}{dt} \frac{d}{du} f(\phi_{-t} \circ \psi_u \circ \phi_t(x)) \\
&= \frac{d}{du} \left(-(\mathcal{L}_X f)(\psi_u) + \mathcal{L}_X(f \circ \psi_u) \right) \\
&= -\mathcal{L}_Y \mathcal{L}_X f + \mathcal{L}_X \mathcal{L}_Y f
\end{aligned}$$

□

The Lie derivative can be extended on tensor fields by compatibility : in all these definitions, we take a vector field X

- If ω is a one form, the Lie derivative of ω in the direction X is defined such that for all vector field Y
$$(\mathcal{L}_X \omega)(y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y)$$
- If $u \in Bil(TM)$, the Lie derivative of u is defined such that for all vector fields Y, Z
$$(\mathcal{L}_X u)(Y, Z) = \mathcal{L}_X(u(Y, Z)) - u(\mathcal{L}_X Y, Z) - u(Y, \mathcal{L}_X Z).$$

- If $A \in \mathcal{L}(TM, TM)$, the Lie derivative of A in the direction X is defined such that for all vector field Y
$$(\mathcal{L}_X A)(Y) = \mathcal{L}_X(A(Y)) - A(\mathcal{L}_X Y).$$

Exercise 1.
One can check that $\mathcal{L}_X \omega = \frac{d}{dt}(\phi_t^* \omega)|_{t=0}$.

The Lie derivative is indeed a derivative, in the sense that it is $\mathbb{R}$ linear, and satisfies the Leibnitz rule
$$\mathcal{L}_X(fY) = f\mathcal{L}_X Y + (X(f))Y.$$
However, there is something unsatisfactory about it which is that the value of $\mathcal{L}_X Y$ at a point p does not depend only on the value of X at p, but also on the behaviour of X near p. This motivate the definitions of the next section.

2.1.7 Connexions

Definition 2.1.25. *A connection D on the vector bundle (E, M) is an application $D : TM \times E \to E$ such that*

- *For all $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$, $Z \in \Gamma(E)$, $D_{fX+Y} = fD_X Z + D_Y Z$,*
- *For all $\lambda \in \mathbb{R}$, $X \in \Gamma(TM)$, $Z, W \in \Gamma(E)$, $D_X(\lambda Z + W) = \lambda D_X Z + D_X W$,*
- *For all $f \in C^\infty(M)$ and $X \in \Gamma(TM)$, $Z \in \Gamma(E)$, $D_X(fZ) = fD_X Z + (\mathcal{L}_X f)Z$.*

If $(x^1, .., x^n)$ is a coordinate chart on M, and $(e_1, .., e_p)$ a local trivialization on E we can write
$$D_{\frac{\partial}{\partial x^i}} e_a = \Gamma^b_{ia} e_b.$$
The Γ^b_{ia} are called the Christoffel symbols of the connection. If $\sigma \in \Gamma(E)$ and $\sigma = \sum \sigma^a e_a$ then
$$D_{\frac{\partial}{\partial x^i}} \sigma = \frac{\partial \sigma^a}{\partial x^i} e_a + \sigma^a \Gamma^b_{ia} e_b.$$

Exemple on a trivial bundle If $M \times \mathbb{R}^p$ is a trivial bundle, and $\sigma \in \Gamma(E)$, we can write $\sigma = (\sigma^1, .., \sigma^p)$. Then
$$D_X \sigma = (\mathcal{L}_X(\sigma_1), .., \mathcal{L}_X(\sigma_p))$$
defines a "trivial" connection on $M \times \mathbb{R}^p$. The Christoffel symbols in this trivialization are zero.

Be careful ! It is not because the Christoffel symbols are zero in some basis that they are zero in every basis ! For instance if we consider the trivial connection on $\mathbb{R}^2$ defined, in Euclidean coordinates (x^1, x^2) by $D_{\frac{\partial}{\partial x^i}}(X^j \frac{\partial}{\partial x^j}) = \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j}$ then the Christoffel symbols in the basis $\frac{\partial}{\partial x^j}$ are zero, but in a local basis defined by $e_r = \frac{\partial}{\partial r}$, $e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}$ where r, θ are the polar coordinates, we have for instance
$$D_{\frac{\partial}{\partial x^i}} e_\theta = D_{\frac{\partial}{\partial x^i}} \left(-\sin(\theta)(\theta) \frac{\partial}{\partial x^1} + \cos(\theta) \frac{\partial}{\partial x^2} \right) = -\frac{\partial \sin(\theta)}{\partial x^i} \frac{\partial}{\partial x^1} + \frac{\partial \cos(\theta)}{\partial x^2} \frac{\partial}{\partial x^i}$$
and consequently
$$D_{e_\theta} e_\theta = -\frac{\partial}{\partial r}.$$

2.2 Riemannian manifolds

2.2.1 Riemannian metric

If E is a vector space, we note Sym^2E the vector space of symmetric bilinear forms on E. We introduce the tensor bundle $Sym^2M = \{(x,q) \in M \times Sym^2(\mathbb{R}^n), x \in M, q \in Sym^2(T_xM)\}$.

Definition 2.2.1. *A metric on a differential manifold M is a section of $\Gamma(Sym^2M)$ which is positive definite on all point. In other word, it is the smooth data of a scalar product on each fibre T_xM. A differential manifold M equipped with a metric g is called a Riemannian manifold.*

In a coordinate chart x^i we can write $g = g_{ij}dx^idx^j$. This means that if $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^j\frac{\partial}{\partial x^j}$ we have $g(X,Y) = g_{ij}X^iX^j$.

Example The metric on the sphere S^2 in the spherical coordinates θ, ϕ is given by

$$g = d\theta^2 + \sin^2(\theta)d\phi^2.$$

The prescription of a metric g allows to identify the tangent space and the cotangent space in the following sense : if X is a vector field, we can define a unique one-form $\omega_X \in \Gamma(T^*M)$ by requiring that for all $Y \in \Gamma(TM)$ we have $\omega_X(Y) = g(X,Y)$. In a coordinate chart x^i we can write $\omega_X = (\omega_X)_jdx^j$ with $(\omega_X)_j = g_{ij}X^j$. We will often denote a vector field $X = X^i\frac{\partial}{\partial x^i}$ by X^i, and a one form $\omega = \omega_jdx^j$ by ω_j : the index up or down then indicate whether we speak about a vector field or a one form. The metric allows to raise and lower indices to make an identification. For instance we will write $(\omega_X)_j = g_{ij}X^j = X_j$.

The metric g induced also a metric on g^{-1} on T^*M, such that for all $X, Y \in \Gamma(TM)$,

$$g(X,Y) = g^{-1}(\omega_X, \omega_Y),$$

which in coordinates can be written $g_{ij}X^iY^j = g^{ij}X_iY_j$.

2.2.2 The Levi-Civita connexion

Proposition 2.2.2. *There exists a unique connection ∇ on TM such that*

- *For all vector fields X, Y we have $\nabla_XY - \nabla_YX = [X,Y]$: we say that ∇ is torsion free.*
- *$\nabla g = 0$, which means that for all vector fields X, Y, Z we have*

$$\mathcal{L}_Xg(Y,Z) = g(\nabla_XY, Z) + g(Y, \nabla_XZ) :$$

we say that ∇ is metric.

This connection ∇ is called the Levi-Civitta connexion.

Proof. **Unicity:** By the property that ∇ is metric we have

$$\mathcal{L}_Xg(Y,Z) = g(\nabla_XY, Z) + (Y, \nabla_XZ)$$
$$\mathcal{L}_Yg(X,Z) = g(\nabla_YX, Z) + (X, \nabla_YZ)$$
$$\mathcal{L}_Zg(X,Y) = g(\nabla_ZX, Y) + (X, \nabla_ZY)$$

Consequently

$$\mathcal{L}_X g(Y,Z) + \mathcal{L}_Y g(X,Z) - \mathcal{L}_Z g(X,Y) = g([X,Z],Y) + g(X,[Y,Z]) + g([Y,X],Z) + 2g(\nabla_X Y, Z)$$

and therefore

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(\mathcal{L}_X g(Y,Z) + \mathcal{L}_Y g(X,Z) - \mathcal{L}_Z g(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])\right).$$

This formula, called the Koszul formula express $g(\nabla_X Y, Z)$ independently on the connection ∇.

Existence: The Koszul formula gives a formula for $g(\nabla_X Y, Z)$. Then it remains to check that this define a connection which is metric and torsion free. This is left to the reader. □

Exercise 2.

By using the Koszul formula, you can express the Christoffel symbols of the Levi-Civita connection in a coordinate chart (x^i).

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}).$$

The Levi-Civita connection can be extended to derive all types of tensor fields by compatibility : in all these definitions, we take a vector field X

- If ω is a one form, $\nabla_X \omega$ is such that for all vector field Y

$$(\nabla_X \omega)(y) = \mathcal{L}_X(\omega(Y)) - \omega(\nabla_X Y).$$

- If $u \in Bil(TM)$ $\nabla_X u$ is such that for all vector fields Y, Z

$$(\nabla_X u)(Y,Z) = \mathcal{L}_X(u(Y,Z)) - u(\nabla_X Y, Z) - u(Y, \nabla_X Z).$$

- If $A \in \mathcal{L}(TM, TM)$, $\nabla_X A$ is such that for all vector field Y

$$(\nabla_X A)(Y) = \nabla_X(A(Y)) - A(\nabla_X Y).$$

In particular, we can apply the Levi-Civita connection to the metric g, which is a section of $Bil(TM)$. The condition that ∇ is metric means that for all $X \in \Gamma(TM)$ we have $\nabla_X g = 0$.

Exercise 3.

In a coordinate system, we can write $\nabla_{\frac{\partial}{\partial x^i}} \omega = (\nabla_{\frac{\partial}{\partial x^i}} \omega)_j dx^j$ with

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \frac{\partial \omega_j}{\partial x^i} - \Gamma^l_{ij}\omega_l.$$

and $\nabla_{\frac{\partial}{\partial x^i}} A = (\nabla_{\frac{\partial}{\partial x^i}} A)^l_j \frac{\partial}{\partial x^l} dx^j$ with

$$(\nabla_{\frac{\partial}{\partial x^i}} A)^l_j = \frac{\partial A^l_j}{\partial x^i} + \Gamma^l_{ik} A^k_j - \Gamma^k_{ij} A^l_k.$$

We will often use a slightly confusing notation and write, for Y a vector field $(\nabla_{\frac{\partial}{\partial x^i}} Y)^j = \nabla_i Y^j$, or for σ a one form $(\nabla_{\frac{\partial}{\partial x^i}} \sigma)_j = \nabla_i \sigma_j$.

2.2.3 Geodesics

In $\mathbb{R}^n$, the straight lines are parametrized curves with a zero acceleration. In (M, g), the acceleration of a path $c : I \to M$ is the derivative of the speed $\dot{c}$ in the direction of the speed $\dot{c}$, in other words $\nabla_{\dot{c}}\dot{c}$. Note that their may be some ambiguity because $\dot{c}$ is only defined on the path $c(t)$: we have $\dot{c}(t) \in T_{c(t)}M$. Therefore $\dot{c}$ is not a vector field. However, if $X \in \Gamma(M)$ we can define $(\nabla_{\dot{c}}X)_{c(t)}$, since the connection have been introduced precisely for this to depend only on the value of $\dot{c}$ on $c(t)$. In coordinates, we have

$$(\nabla_{\dot{c}}X^j)_{c(t)} = (\dot{c}^i\partial_i X^j)(c(t)) + \Gamma^j_{ik}X^k(c(t))\dot{c}^i = \frac{d}{dt}(X^j(c(t))) + \Gamma^j_{ik}X^k(c(t))\dot{c}^i.$$

We see that in addition, this quantity depends only on the value of X on $c(t)$. Therefore $\nabla_{\dot{c}}\dot{c}$ is well defined.

Definition 2.2.3. *A geodesic is a path $c : I \to M$ such that $\nabla_{\dot{c}}\dot{c} = 0$, where ∇ is the Levi-Civita connection. More generally, we will say that a vector field X is geodesic if $\nabla_X X = 0$.*

In coordinates, if we write $c(t) = (x^1(t), .., x^2(t))$ then the geodesic equation is

$$\ddot{x^i} + \Gamma^i_{jk}\dot{x^j}\dot{x^k} = 0.$$

This is a second order non linear equation. : given $p_0 \in M$ and an initial speed $v_0 \in T_xM$ there exists a unique geodesic, defined on a maximal interval $I \subset \mathbb{R}$ such that $c(0) = p_0$ and $\dot{c}(0) = v_0$.

2.2.4 Curvature

The curvature tensor is an object which measures the defect of commutation of connexions. It can be defined for any connexion, but let us focus on the case of the Levi-Civita Connection.

Definition 2.2.4. *Let (M, g) be a Riemannian metric. Let ∇ be the Levi-Civita connexion. Riem is the section of the tensor bundle of anti-symmetric bilinear maps from TM to $End(TM)$ defined for all $X, Y, Z \in \Gamma(TM)$ by*

$$Riem(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

This definition is in fact also a proposition because it is not clear a priori that *Riem* defined above is indeed a tensor. For this we have to check that *Riem* is C^∞ linear with respect to X, Y and Z. Let us check the C^∞ linearity with respect to X :

$$\begin{aligned}Riem(fX, Y)Z =& f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z\\ =& f\nabla_X\nabla_Y Z - f\nabla_Y(\nabla_X Z) - Y(f)\nabla_X Z - f\nabla_{[X,Y]-}Z + Y(f)\nabla_X Z\\ =& fRiem(X, Y)Z\end{aligned}$$

Exercise 4.

In a coordinate system, we can write $Riem(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\mu})\frac{\partial}{\partial x^\beta} = R^\alpha{}_{\beta\gamma\mu}\frac{\partial}{\partial x^\alpha}$, with

$$R^\alpha{}_{\beta\gamma\mu} = \frac{\partial}{\partial x^\gamma}\Gamma^\alpha_{\mu\beta} - \frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\gamma\nu}\Gamma^\nu_{\mu\beta} - \Gamma^\alpha_{\mu\nu}\Gamma^\nu_{\gamma\beta}.$$

Proposition 2.2.5. *The Riemann tensor satisfies the following properties :*

- $Riem(X,Y) = -Riem(Y,X)$
- $g(Riem(X,Y)Z,T) = -g(Riem(X,Y)T,Z)$
- $Riem(X,Y)Z + Riem(Y,Z)X + Riem(Z,X)Y = 0$ *: this property is called Bianchi's first identity.*
- $g(Riem(X,Y)Z,T) = g(Riem(Z,T)X,Y)$
- $(D_Z R)(X,Y)W + (D_X R)(Y,Z)W + (D_Y R)(Z,X)W = 0$ *: this property is called Bianchi's second identity.*

Remark 2.2.6. *These properties ensure that if M is $2-$dimensional, the curvature is determined by a number in each point x : $g(R(e_1,e_2)e_2,e_1)$ where (e_1,e_2) is an orthonormal basis of T_xM. This number is called Gauss curvature.*

Theorem 2.2.7. *Riemann, 1854 Let (M,g) be a Riemannian manifold. Then $Riem = 0$ if and only if (M,g) is locally isometric to $(\mathbb{R}^n,\delta)$ where δ is the Euclidean metric (this means that in the neighbourhood of every points, there exists coordinates $x^1,..,x^n$ such that $g = \sum(dx^i)^2$.*

Geodesic deviation Their is a geometric interpretation of the Riemann curvature tensor. Let $\gamma_s(t)$ be a one parameter family of geodesics, such that $(s,t) \mapsto \gamma_s(t)$ define a submanifold Σ of M. The vector field $T = \frac{\partial}{\partial t}$ is tangent to the geodesics. Up to a reparametrization of t, we can assume $\nabla_T T = 0$. We note $X = \frac{\partial}{\partial s}$ the infinitesimal displacement from a geodesic to the other. Then $v = \nabla_T X$ is the speed of the displacement, and $a = \nabla_T v$ its acceleration. Since s and t are coordinates, we have $[X,T] = 0$, and we can compute

$$a = \nabla_T\nabla_T X = \nabla_T\nabla_X T = Riem(T,X)T + \nabla_X\nabla_T T = Riem(T,X)T.$$

Definition 2.2.8. *The Ricci tensor is the element of $\Gamma(Sym^2(TM))$ defined by*

$$Ric(X,Y) = Tr(Z \mapsto Riem(Z,X)Y).$$

In a coordinate chart

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}.$$

We can contract one time more and define the scalr curvature, which is the function R is defined by $R = g^{ij}R_{ij}$.

2.2.5 Second fundamental form

Let Σ be a submanifold of M of dimension $n-1$, oriented by a unit normal vector N. The metric g on M induce a metric $\bar{g}$ on Σ : two vectors $X,Y \in T_x\Sigma$ can be seen as vectors on T_xM and one can define $\bar{g}(X,Y) = g(X,Y)$. This induced metric induces a connection $\bar{\nabla}$ on Σ. We define the second fundamental form $K \in \Gamma(Sym^2(T\Sigma))$ by : for all X,Y vector fields in $\Gamma(T\Sigma)$

$$\nabla_X Y = \bar{\nabla}_X Y + \mathrm{II}(X,Y)N.$$

In other words

$$\mathrm{II}(X,Y) = g(\nabla_X Y, N).$$

One has to check that K is indeed a tensor, and is symmetric (exercise).

Proposition 2.2.9. *The second fundamental form is also given by* $\mathrm{I\!I} = -\frac{1}{2}\mathcal{L}_N g$ *: for all* X, Y *tangent to* Σ *we have*

$$\mathrm{I\!I}(X,Y) = -\frac{1}{2}(\mathcal{L}_N g)(X,Y).$$

Proof. We start with a general computation : for all vector fields X, Y, Z we have

$$\begin{aligned}(\mathcal{L}_X g)(Y,Z) =& \mathcal{L}_X(g(Y,Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z)\\ =& \mathcal{L}_X(g(Y,Z)) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X)\\ =& g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\end{aligned}$$

where in the first equality, we have used the definition of the Lie derivative of a 2-tensor, in the second equality we have used that the Levi-Civita connection is torsion free, and in the last equality we have used the fact that the Levi-Civita connection is metric, that is to say $\nabla g = 0$. Now, if we use the definition of the second fundamental form, we write, for X, Y vector fields tangent to Σ

$$\begin{aligned}\mathrm{I\!I}(X,Y) =& g(\nabla_X Y, N)\\ =& \frac{1}{2}g(\nabla_X Y + \nabla_Y X, N) + \frac{1}{2}g([X,Y],N)\\ =& \frac{1}{2}\left(\mathcal{L}_X(g(Y,N)) - g(Y,\nabla_X N) - \mathcal{L}_Y(g(X,N)) - g(X,\nabla_Y N)\right)\\ =& -\frac{1}{2}(\mathcal{L}_N g)(X,Y),\end{aligned}$$

where we have used that $[X,Y]$ is tangent to Σ and therefore $g([X,Y],N) = g(X,N) = g(Y,N) = 0$. ☐

2.2.6 Symmetries

Definition 2.2.10. *Let* $\phi : U \subset M \to V \subset M$ *be a differomorphism. The pull-back of a 2-form* u *in* $T_{\phi(p)}M$ *is the 2-form in* T_pM *defined by*

$$(\phi^* u)_p(X,Y) = u_{\phi(p)}(d\phi X, d\phi Y).$$

Definition 2.2.11. *A diffeomorphism* $\phi : U \subset M \to M$ *is an isometry if*

$$\phi^* g = g.$$

A vector field whose one parameter flow is a flow of isometries is called a Killing field.

Proposition 2.2.12. K *is a Killing field on* (M, g) *if and only if*

$$\mathcal{L}_K g = 0.$$

Proof. Let ϕ_t be the one parameter flow generated by K.

$$(\phi_t^* g)(X,Y)|_p = g(d\phi_t X, d\phi_t Y)|_{\phi_t(p)} = g(\phi_{-t}^* X, \phi_{-t}^* Y)|_{\phi_t(p)},$$

Consequently, by definition of the Lie derivative we have

$$\frac{d}{dt}\left[(\phi_t^* g)(X,Y)|_p)\right]_{t=0} = K(g(X,Y)) - g(\mathcal{L}_K X, Y) - g(X, \mathcal{L}_K Y) = (\mathcal{L}_K g)(X,Y).$$

Consequently, if K is a Killing field, which means that for all t, ϕ_t is an isometry then $(\phi_t^* g)(X,Y) = g(X,Y)$ and we have $(\mathcal{L}_K g)(X,Y) = 0$.

Conversely, if $\mathcal{L}_K g = 0$, using the fact that $\phi_{t+t'} = \phi_t \circ \phi_{t'}$ we have

$$\frac{d}{dt}\left[(\phi_t^* g)(X,Y)\right] = 0,$$

so $(\phi_t^* g)(X,Y) = (\phi_0^* g)(X,Y) = g(X,Y)$ and ϕ_t is an isometry for all t. □

Exercise 5.

Let X be a vector field. Show that $\mathcal{L}_X g = {}^{(X)}\pi$ where ${}^{(X)}\pi$, the deformation tensor of X is defined by

$${}^{(X)}\pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha.$$

A vector field K is Killing if and only if ${}^{(X)}\pi = 0$. This is something non generic for a space-time to have Killing fields. Indeed, ${}^{(X)}\pi = 0$ is a system of $\frac{n(n+1)}{2}$ equations with n unknowns.

2.2.7 Integration

Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension n. In a coordinate system, one can define $det(g)$ as the determinant of the matrice $g_{\alpha\beta}$. This does not define a function on the manifold $\mathcal{M}$ since this is a coordinate dependant quantity. However, we have the following property :

Lemma 2.2.13. *Let $(\mathcal{U}, (x^1, .., x^n))$ be a chart on $\mathcal{M}$, and $f : \mathcal{M} \to \mathbb{R}$ a scalar function. Then*

$$\int_{\mathcal{U}} f\sqrt{det(g)}dx^1..dx^n$$

is invariant by change of coordinates.

Proof. Let $y^i(x^1, .., x^n)$ be a change of coordinate. Doing a change of variable in the integral, we obtain

$$\int_{\mathcal{U}} f\sqrt{det(g)}dx^1..dx^n = \int_{\mathcal{U}} f\sqrt{det(g)}\left|\frac{\partial x}{\partial y}\right| dy^1..dy^n,$$

where $\left|\frac{\partial x}{\partial y}\right|$ is the Jacobian, that is to say the determinant of the matrix of the change of coordinates. If we note $det(\tilde{g})$ the determinant of the matrix $\tilde{g}_{\alpha\beta}$ of the metric coefficients in the basis $dy^\alpha dy^\beta$, we have

$$det(\tilde{g}) = \left|\frac{\partial x}{\partial y}\right|^2 det(g).$$

This concludes the proof of the lemma. □

We can now define the integration on $\mathcal{M}$ with the use of a partition of unity ψ_α adapted to a covering by coordinate charts $\mathcal{U}_\alpha$.

$$\int_{\mathcal{M}} f dvol_g = \sum_\alpha \int_{\mathcal{U}_\alpha} \psi_\alpha f\sqrt{det(g)}dx^1..dx^n.$$

Exercise 6.

We have, in a coordinate chart

$$\int_{\mathcal{U}} (D^\alpha X_\alpha) dvol_g = \int \partial_\alpha X^\alpha dx.$$

2.3 Exercises

Exercise 7.

1. Let us consider the chart on $\mathbb{S}^2$ given by (U_i, ϕ_i), with $i = 1..6$ where for $1 \le i \le 3$, $U_i = \{x \in \mathbb{S}^2,\ x_i > 0\}$, $U_{i+3} = \{x \in \mathbb{S}^2,\ x_i < 0\}$ and

$$\phi_1(x) = \phi_4(x) = (x^2, x^3),\ \phi_2(x) = \phi_5(x) = (x^1, x^3),\ \phi_3(x) = \phi_6(x) = (x^1, x^2).$$

Check on a few changes of charts that this atlas is equivalent to the stereographic projection.

2. Can you build yet another atlas on $\mathbb{S}^2$?

Exercise 8.

1. Is the subset of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = 0$ a submanifold of $\mathbb{R}^3$?

2. Is the map $\mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^2, t^3)$ an immersion ? Show that its image is not a submanifold of $\mathbb{R}^2$.

3. Show that the map $]-\infty, 1[\to \mathbb{R}^2$ defined by $t \mapsto \left(\frac{t^2-1}{t^2+1}, \frac{t(t^2-1)}{t^2+1}\right)$ is an injective immersion, but that its image is not a submanifold of $\mathbb{R}^2$. Draw the image.

4. Show that the group $SL(n, \mathbb{R})$ is a submanifold of the vector space of matrices $n \times n$. What is its dimension ? Same question with $O(n)$.

Exercise 9.

Let U be an open set of $\mathbb{R}^n$, $a \in U$ and $f : U \to \mathbb{R}^p$. If f is a submersion, show that the tangent space to $f^{-1}(f(a))$ in a is the kernel of df_a. If f is an immersion, show that the tangent space to $f(U)$ in $f(a)$ is the image of df_a.

Exercise 10.

We say that two metrics g and g' on M are conformal if there exists a function $f \in C^\infty(M)$ such that $g' = e^f g$. Let D and D' be respectively the Levi-Civita connection associated to g and g'. Show that

$$2D'_X Y = 2D_X Y + X(f)Y + Y(f)X - g(X,Y) grad_g(f)$$

where $grad_g(f)$ is the unique vector field such that for all $X \in \Gamma(TM)$, $X(f) = g(grad_g(f), X)$.

Exercise 11.

Let D be a connexion on a vector bundle E with basis M. Let $c : I \to M$ be a smooth curve.

1. Show that $(D_{\dot c} s)_{c(t)}$ depends only on s on the curve c.

Given a curve c, we define the parallel transport $P^c : E_{c(a)} \to E_{c(b)},\ s(a) \mapsto s(b)$ where $s(t)$ is the solution to $D_{\dot c} s = 0$.

2. Show that $D_X\sigma = \frac{d}{dt}(\tilde{\sigma}(t))$ where we took any curve c with $c(0) = x$ and $\dot{c}(0) = X_x$ and define $\tilde{\sigma}(t) = (P^c)^{-1}(\sigma(c(t)))$.

3. We now consider the Levi-Civitta connexion ∇. Let $X_x, Y_x \in T_xM$ be such that $g_x(X_x, Y_x) = 0$. Let c be a path from x to y. Show that

$$g(P^cX, P^cY) = 0.$$

Exercise 12.

Let $\mathcal{M}$ be an hypersurface of $\mathbb{R}^3$ (a submanifold of dimension 2). We consider $\mathbb{R}^3$ with the Euclidean metric. Let D the flat connection on $\mathbb{R}^3$ and ∇ the induced connection on the sphere.

1. Let $X, Y \in TM$. Show that $(D_XY)_p - (\nabla_XY)_p$ is colinear to the normal to M at p.
2. Show that $\gamma(s)$ is a geodesic on M if and only if $\ddot{\gamma}$ is colinear to the normal.
3. Characterise the geodesics on the sphere.

Exercise 13.

Let (M, g) be a Riemannian manifold, and γ be a geodesic. Suppose that their exists a Killing field K. Show that $g(\dot{\gamma}, K)$ is constant along γ.

Exercise 14.

Let (M, g) be a Riemannian manifold, and D the Levi-Civita connection associated to g.

1. Show that $(\mathcal{L}_Kg)_{\alpha\beta} = D_\alpha K_\beta + D_\beta K_\alpha$.
2. Assume that K be a Killing field. Show that

$$D_XD_YK = -Riem(K, X)Y.$$

Exercise 15.

Let (M, g) be a Riemannian manifold, and $\mathcal{H} \subset M$ an hypersurface , with the induced metric. Let $\mathrm{I\!I}$ be the second fundamental form. Let $\mathbf{R}$ be the Riemannian curvature tensor on M and R the Riemannian curvature tensor on $\mathcal{H}$.

1. Show the Gauss theorem

$$\mathbf{R}_{ijlm} = R_{ijlm} - \mathrm{I\!I}_{mj}\mathrm{I\!I}_{li} + \mathrm{I\!I}_{lj}\mathrm{I\!I}_{mi}.$$

Write the corresponding relation for the Ricci tensor and the scalar curvature.

2. Calculate the second fundamental form of the embedding of $\mathbb{S}^n$ in $(\mathbb{R}^{n+1}, \delta)$. Deduce the Riemann curvature, the Ricci curvature and the scalar curvature of the sphere.

Chapter 3

Lorentzian geometry and general relativity

3.1 The geometry of space-time in special relativity

In Newton mechanics, space-time is a direct product $\mathbb{R}_t \times \mathbb{R}^3$, where $\mathbb{R}_t$ is the set of times, and the space is $\mathbb{R}^3$ equipped with the Euclidean metric $\delta = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. The events at $t = cst$ are simultaneous. The future of a point $p = (0, x)$ is $t > 0$ and the past $t < 0$. Newton's laws are invariant under

- The symmetries of the Euclidean space : translations and rotations,
- The time translations,
- The Galileo transform $(x^i)' = x^i + v^i t$.

In particular, the speed can not be an absolute value, it depends on the choice of inertial frame. This leads to contradiction with the fact that the speed of light is a physical constant. This is indeed a prediction of Maxwell equations, which unify the laws governing the electric field E and the magnetic field B (written here with "units" such that $\varepsilon_0 = \mu_0 = 1$)

$$\begin{aligned}
\nabla \wedge E &= -\frac{\partial B}{\partial t}, \\
\nabla \wedge B &= j + \frac{\partial E}{\partial t}, \\
\nabla . E &= \rho, \\
\nabla . B &= 0.
\end{aligned}$$

Maxwell's laws are not left invariant by the Galileo transform. Instead they are invariant by the following transformations

- The translations in space or time,
- The space rotations,

- The hyperbolic rotations

$$L_i : \begin{array}{ll} t & \to \cosh(\alpha)t + \sinh(\alpha)x^i \\ x^i & \to \sinh(\alpha)t + \cosh(\alpha)x^i \end{array} .$$

These transformations generate a group, discovered by Poincaré and Lorentz, which is also the group which leaves invariant the quadratic form

$$m = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{3.1.1}$$

The quadratic form m is called Minkowski metric. It is of signature $(-1, 1, 1, 1)$. In special relativity, the laws of physics are independent of the choice of inertial coordinates, which are the coordinates in which m can be written like (3.1.1). For instance, if (t, x^i) are inertial coordinates, so are $(t', (x^1)', x^2, x^3)$ where $t', (x^1)'$ are obtained through an hyperbolic rotation of parameter α. In particular, the time coordinate t is not an absolute quantity any more !

When $\mathbb{R}^4$ is equipped with the quadratic form m, one can make a distinction between vectors in $\mathbb{R}^4$ in the following way.

- X is timelike if $m(X, X) < 0$
- X is light-like if $m(X, X) = 0$
- X is space-like if $m(X, X) > 0$.

An observer corresponds to a curve in space-time which is causal, meaning that its tangent vector is always time-like or light-like. Let p be a point of space-time. We can consider all the points which can be joined with a causal curve. It has two components : one is the future of p, and the other the past of p.

We can define the proper time of an observer $\gamma : [a, b] \to \mathbb{R}^4$ between $\gamma(a)$ and $\gamma(b)$ by

$$\int_a^b \sqrt{-m(\dot\gamma(s), \dot\gamma(s))} ds.$$

3.1.1 The Poincaré group

Minkowski metric admits the following symmetries :

- The translations $x \mapsto x + a$ for $a \in R^{1+3}$,
- The transformations $x \mapsto Ax$, where A is a 4×4 matrix such that $m(Ax, Ay) = (x, y)$: these transformations are called Lorentz transforms. One can make a distinction between space rotations and hyperbolic rotations.

The group generated by these isometries is called the Poincaré group. These isometries correspond to Killing fields which can be written in inertial coordinates (x^0, x^1, x^2, x^3), with $x^0 = t$ in the following way

- for the generators of translations : $\frac{\partial}{\partial x^\alpha}$,
- for the generators of space rotations : $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$, $1 \leq i < j \leq 3$,
- for the generators of hyperbolic rotations : $x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}$, $1 \leq i \leq 3$.

3.1.2 Maxwell equations

Their is a covariant way of expressing Maxwell equations. In inertial coordinates (t, x^1, x^2, x^3), we write

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

F can be seen as a 2 form which is antisymmetric. Maxwell equations can be written

$$D_\alpha F_{\beta\gamma} + D_\gamma F_{\alpha\beta} + D_\beta F_{\gamma\alpha} = 0,$$
$$D^\alpha F_{\alpha\beta} = J_\beta,$$

where J_β is the source term.

3.2 The geometry of space-time in General Relativity

Once the theory of special relativity had been constructed, the next task had been to reformulate physicals law in this setting. However, writing Newton's law of universal gravitation in the context of special relativity was a special challenge. Indeed, it invokes a concept of "action at a distance", incompatible with the causality notions of special relativity. One of the principle which lead to Geneneal relativity is the equivalence principle, according to which all bodies are influenced in the same way by the gravitational field. The path of freely falling bodies define a preferred set of curves in space-time, just as in special relativity did the paths of inertial bodies. This paths are the geodesics of the space-time. In general relativity, the gravitational field corresponds to a deviation of the space-time geometry from the flat geometry of special relativity. More precisely, the space-time of General Relativity is described by a Lorentzian manifold $(\mathcal{M}, g)$ that we introduce now.

3.2.1 Lorentzian manifolds

Definition 3.2.1. *A Lorentzian metric on a manifold M is a section of $Sym^2 M$ which is everywhere non degenerate and of signature* $(-1, 1, 1, 1)$.

A Lorentzian manifold is a manifold M equipped with a Lorentzian metric g. All the notions of Section 2.2 can be defined in the context of Lorentzian manifolds : connections, geodesics, curvature... Moreover, with a Lorentzian metric comes a new notion which is causality.

A vector $v \in T_x M$ is called :

- spacelike if $g_x(v, v) > 0$,
- null or light-like if $g_x(v, v) = 0$,
- timelike if $g_x(v, v) > 0$,
- causal if $g_x(v, v) \geq 0$.

The set of causal vectors in T_xM is called the causal cone, noted C_x, and its boundary, the set of null vectors is called the light cone.

One can separate the causal cone $C_x \setminus \{0\}$ into two connected components C_x^+ and C_x^-. If it is possible to make a continuous choice of C_x^+ and C_x^- with respect to $x \in M$, we will say that the manifold is time oriented.

Let $\gamma : [a,b] \to M$ be a parametrized curve. It is

- light-like or null if for all $s \in [a,b]$, $\dot{\gamma}(s)$ is null (future null if in addition, for all s, $\dot{\gamma}(s) \in C_x^+$),
- timelike if for all $s \in [a,b]$, $\dot{\gamma}(s)$ is timelike (future timelike if in addition, for all s, $\dot{\gamma}(s) \in C_x^+$),
- causal if for all $s \in [a,b]$, $\dot{\gamma}(s)$ is causal (future causal if in addition, for all s, $\dot{\gamma}(s) \in C_x^+$).

An hypersurface Σ of M is called space-like if all its tangent vectors are space-like. The restriction of the metric g to Σ is then a Riemannian metric.

3.2.2 Einstein equations

In Newton's theory, the gravitational field is the gradient of the Newtonian potential U, and the motion of a test particle in the gravitational field obeys a differential equation, independent on its mass $\ddot{x}^i = \frac{\partial U}{\partial x^i}$. Moreover, the potential U satisfies the Poisson equation $\Delta U = -4\pi\kappa\rho$, where ρ is the mass density of the source and κ is the Newtonian gravitational constant. If we compare the equation of motion with the geodesic equation

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0,$$

we see that the "equivalent" of the gravitational potential in general relativity should be the metric itself. Einstein equations link the metric g to the source. They can be written

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci tensor of g, R is the scalar curvature, Λ the cosmological constant, and $T_{\mu\nu}$ the stress-energy tensor. Let us make some comments on these equations

- These equations obey the principle of general covariance : the physical phenomena do not depend on the reference frame in which we express their laws.
- Like Newton's equations, these equations are second order in g.
- One often says that the space-time is curved in the presence of matter. The first object we have introduced in the course to describe the curvature is the Riemann tensor. However one could not ask that in the absence of source, $Riem = 0$ because the effects of gravitation can be felt far from the sources !
- The stress-energy tensor was already an object introduced in special relativity to describe matter distributions. If an observer is described by a four velocity v^α, then $T_{\alpha\beta}v^\alpha v^\beta$ is the energy mesured by the observer. From the other components of $T_{\alpha\beta}$, one can compute the linear momentum, the stress...

Proposition 3.2.2. *Let (M, g) be a Riemannian or Lorentzian manifold. We have*

$$D^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0.$$

These identities are called the contracted Bianchi identities.

This proposition implies that Einstein equations yield the following identities for T

$$D^\mu T_{\mu\nu} = 0.$$

These identities are called local conservation laws.

3.2.3 Examples of Energy impulsion tensors

The form of the energy impulsion depend on the matter model under study. Let us give two examples.

- If the matter is described by a perfect fluid, with 4 velocity u, density of energy ρ, and pressure P

  $$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu).$$

  The local conservation laws $D^\mu T_{\mu\nu} = 0$ yield Euler's equations.

- If there is an electromagnetic field, given by an antisymmetric 2 form F then

  $$T_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}.$$

  When there are no sources for Maxwell equations, $D^\mu T_{\mu\nu} = 0$ yields Maxwell equations. If there are sources, one should add the energy impulsion tensor of a charged fluid for instance.

3.2.4 Some explicit solutions

We consider first the vacuum case, that is to say $T_{\mu\nu} = 0$, with zero cosmological constant. In that case, by taking the trace with respect to the metric g of Einstein equations, one obtain (in dimension $3 + 1$)

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = R - 2R = 0,$$

so the scalar curvature, and consequently the Ricci tensor vanishes. A very special solution is Minkowski metric, the metric of Special Relativity. The Riemann tensor of Minkowski metric vanishes and so does a fortiori the Ricci tensor. However, as was said before, their are solutions to Einstein vacuum equations for which the Riemann tensor is not zero. The first one which was discovered in this category was Schwartzschild solution, whose aim was to describe the gravitation created by a static spherically symmetric star. There is a simple expression for this metric, which is

$$g_S = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

We note that the vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ are Killing fields for the metric.

If we consider the exterior of a star of radius $r_0 > 2M$, the metric g_S is well defined. Let us now consider the manifold $\mathcal{M} = \mathbb{R}\times]2M, +\infty[\times\mathbb{S}^2$ equipped with the metric g_S. It turns out that $\mathcal{M}$ can be isometrically embedded in a larger Lorentzian manifold $(\mathcal{N}, \tilde{g})$, for which the region $r = 2M$ is not singular anymore. This means that $r = 2M$ is only a coordinate singularity.

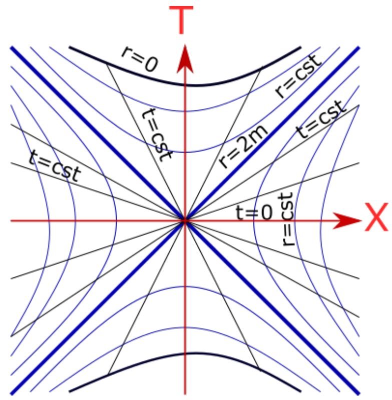

Figure 3.1: The maximal extension of the Schwarzschild space-time : in this figure, each point is a $2-$sphere of radius r.

3.2.5 Toward the study of general solutions

In the following of the course, we want to initiate a study of general solutions to Einstein equations in vacuum. To do this, we will see Einstein equations as evolution partial differential equations. To this extent, we introduce a $3+1$ decomposition of space-time. We assume that the manifold M can be decomposed in $\Sigma \times \mathbb{R}$, where Σ is a three dimensional manifold. We introduce the splitting of the metric

$$g = -N^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

then $\bar{g}_{ij}$ is the Riemanian metric induced on the hypersurface of constant t. N is called the lapse, and β the shift. Let T be the vector field of normals to Σ. We have

$$T = \frac{1}{N}(\partial_t - \beta) = \frac{1}{N} e_0.$$

We also introduced the second fundamental form K.

$$K_{ij} = - <\nabla_{e_i} T, e_j> = -\frac{1}{2N}(\partial_t g_{ij} - \mathcal{L}_\beta g_{ij}).$$

The Ricci tensor can be expressed in term of $N, \beta, \bar{g}$ and K :

Proposition 3.2.3. *Decomposition of the Ricci tensor associated to g:*

$$R_{ij} = \bar{R}_{ij} + K_{ij} K_l{}^l - 2K_i{}^l K_{jl} - N^{-1}(\mathcal{L}_{e_0} K_{ij} + \nabla_i \partial_j N), \tag{3.2.1}$$

$$R_{0j} = N(\partial_j K^h{}_h - \nabla_{\bar{g}} K^h{}_j), \tag{3.2.2}$$

$$R_{00} = N(\partial_0 (K^h{}_h) - N K_{ij} K^{ij} + \Delta N), \tag{3.2.3}$$

Scalar curvature

$$R = \bar{R} + (\bar{g}^{ij} K_{ij})^2 + K^{ij} K_{ij} - 2N^{-1} \partial_0 (K^h{}_h) - 2N^{-1} \Delta N. \tag{3.2.4}$$

Proof. We give only the proof of the first equality. We begin by proving the Theorem Egregium of Gauss

$$\begin{aligned}
R_{ijkl} &= <\nabla_i \nabla_j e_l - \nabla_j \nabla_i e_l, e_k> \\
&= <\nabla_i (\nabla_j e_l - K_{lj} T) - \nabla_j (\nabla_i e_l - K_{il} T), e_k> \\
&= \bar{R}_{ijkl} - K_{lj} <\nabla_i T, e_k> + K_{il} <\nabla_j T, e_k> \\
&= \bar{R}_{ijkl} + K_{lj} K_{ik} - K_{il} K_{jk}.
\end{aligned}$$

and consequently

$$\begin{aligned}
R_{ij} &= g^{\mu\nu} R_{\mu i \nu j} \\
&= g^{lk}(\bar{R}_{likj} + K_{ij} K_{lk} - K_{ik} K_{lj}) + g^{00} N (\mathcal{L}_{e_0} K_{ij} + N K_i{}^l K_{lj} + \nabla_i \partial_j N) \\
&= \bar{R}_{ij} + K_{ij} K_l{}^l - K_i{}^l K_{jl} - N^{-1} (\mathcal{L}_{e_0} K_{ij} + N K_i{}^l K_{lj} + \nabla_i \partial_j N).
\end{aligned}$$

□

We look at the time derivatives in the system: they appear with K, and with $\mathcal{L}_{e_0} K$. This lead to the following choice of initial data:

Initial data

The initial data for Einstein equation are a triplet $(\Sigma, \bar{g}, K)$ with

- Σ a 3-dimensional manifold
- $\bar{g}$ a Riemannian metric on Z
- K a symmetric 2-tensor

Solving Einstein equations with these data consist in finding $(\mathcal{M}, g)$ such that

$$\Sigma \subset \mathcal{M}, \quad g|_\Sigma = \bar{g},$$

and K is the second fundamental form of the embedding of Σ in $\mathcal{M}$.

Evolution equations

There is a gauge freedom in Einstein equations, corresponding to the invariance by diffeomorphism. A natural gauge choice consist in fixing β and N. For example $\beta = 0$, $N = 1$. An other choice, which is often used are wave coordinates, which will be seen more in detail in the end of the course, but which we already present here to motivate our study of wave equations.

We write the Ricci tensor in a coordinate system

$$R_{\nu\nu} = -\frac{1}{2} \Box_g g_{\mu\nu} + \frac{1}{2} (g_{\mu\rho} \partial_\nu H^\rho + g_{\nu\rho} \partial_\mu H^\rho) + P_{\mu\nu}(g)(\partial g, \partial g),$$

where $H^\rho = \Box_g x^\rho = \frac{1}{\sqrt{|det(g)|}} \partial_\alpha (g^{\alpha\rho} \sqrt{|det(g)|})$, and $\Box_g$ is defined by

$$\Box_g u = \nabla^\alpha \nabla_\alpha u = \frac{1}{\sqrt{|det(g)}} \partial_\alpha (\sqrt{|det(g)} g^{\alpha\beta} \partial_\beta u). \tag{3.2.5}$$

In Minkowski metric, the operator $\Box_g$ is the d'Alembertian

$$\Box_g u = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Wave coordinates consist in setting $H^\rho = 0$. They can be seen as the equivalent of Lorenz gauge for Maxwell. In wave coordinates we have

$$R_{\mu\nu} = R^H_{\mu\nu} = -\frac{1}{2}\Box_g g_{\mu\nu} + P_{\mu\nu}(g)(\partial g, \partial g),$$

so Eintein equations can be written as a system of non linear wave equations.

3.3 Exercises

Exercise 1.

1. Let E, B be the electric and magnetic field, solution of the Maxwell equations in vacuum. Let us introduce the 2 tensor on Minkowski space-time

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

Show that Maxwell equations for E and B are equivalent to the equations for $F_{\alpha\beta}$:

$$D_\alpha F_{\beta\gamma} + D_\gamma F_{\alpha\beta} + D_\beta F_{\gamma\alpha} = 0, \tag{3.3.1}$$
$$D^\alpha F_{\alpha\beta} = 0. \tag{3.3.2}$$

2. In a curved space-time (M, g), the Maxwell field is given by a $(0, 2)$ tensor $F_{\alpha\beta}$ and Maxwell equations are (3.3.1) and (3.3.2) with D the Levi-Civita connection associated to g. The stress energy tensor is given by

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{de}F^{de}$$

Show that $D^a T_{ab} = 0$.

Exercise 2.

We consider a metric g which can be written in a coordinate system (t, r, θ, ϕ)

$$g = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{3.3.3}$$

1. Compute the Christoffel symbols of g.
2. Compute the Ricci tensor of g.

Exercise 3.

We consider the metric given for $r > 2M$ by

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

1. Show that it is a solution of Einstein vacuum equation.

2. Let $r^* = r + 2m\ln(r-2m)$, $v = t + r^*$ and $w = t - r^*$. Express the metric g in the coordinates v, w, θ, ϕ.

3. Let $v' = \exp(\frac{v}{4m})$ and $w' = -\exp(-\frac{w}{4m})$. Express the metric g in the coordinates v', w', θ, ϕ and show that the Schwarzchild solution can be extended for $0 < r \leq 2m$.

Exercise 4.

We are now looking for static spherically symmetric solution of vacuum Einstein-Maxwell solutions, with g of the form (3.3.3) and $F_{\alpha\beta}$ such that the only non zero coefficients are $F_{01}(r) = -F_{10}(r)$.

1. Show that their exists solutions under this ansatz, and that their exists M, Q such that $A = B^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

2. Study the singularities of A : are they coordinate singularities or curvature singularities ?

Exercise 5.

In this exercise, we want to study the timelike geodesic in Schwarzschild space-time. We consider a geodesic, parametrized by some parameter s : $x(s) = (t(s), r(s), \theta(s), \phi(s))$.

1. Recall the geodesic equation. Write the second order equation for $\theta(s)$, and explain why, without loss of generality, we can assume that $\theta(s)$ is constant, equal to $\frac{\pi}{2}$.

2. Recall why, for K a Killing vector field, $g(\dot{x}, K)$ is constant along the geodesic. Show the two conservation laws

$$E = \left(1 - \frac{2m}{r}\right)\frac{dt}{ds}, \qquad L = r^2\frac{d\phi}{ds},$$

where E and L are some constants depending on the geodesic.

3. We choose a parametrization where $g(\dot{x}(s), \dot{x}(s)) = -1$. Show that

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L}{r^2} + 1\right) = \frac{1}{2}E^2.$$

4. Discuss the possible trajectories of the geodesics.

Exercise 6.

1. Prove the second Bianchi identity,

$$(D_Z R)(X,Y)W + (D_X R)(Y,Z)W + (D_Y R)(Z,X)W = 0$$

which can be written in coordinates

$$D_a R_{bcde} + D_b R_{cade} + D_c R_{abde} = 0.$$

2. Prove the so called contracted Bianchi identities $D^\mu(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) = 0$.

3. Assume that the Ricci tensor of (M, g) is vanishing. Prove that the Riemann tensor satisfies the following equation

$$D^a D_a R_{bcde} + 2R_c{}^{fa}{}_b R_{fade} + 2R_d{}^{fa}{}_b R_{cafe} + 2R_e{}^{fa}{}_b R_{cadf} = 0$$

Exercise 7.

We study solutions of vacuum Einstein equations of the form $g_{\alpha\beta} = m_{\alpha\beta} + \gamma_{\alpha\beta}$ where $\gamma_{\alpha\beta}$ is small.

1. Show that the linearization of the Ricci tensor around m is given by

$$\delta R_{\alpha\beta} = -\frac{1}{2}\partial^\mu \partial_\mu \gamma_{\alpha\beta} + \frac{1}{2}\partial^\mu (\partial_\alpha \gamma_{\beta\mu} + \partial_\beta \gamma_{\alpha\mu}) - \frac{1}{2}\partial_\alpha \partial_\beta \gamma.$$

where $\gamma = g^{\alpha\beta}\gamma_{\alpha\beta}$.

2. Deduce the linearization of the Einstein tensor.

3. Show that it is invariant by the transformation $\gamma_{\alpha\beta} \to \gamma_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha$ for all vector field ξ (these transformations correspond to infenitesimal difeomorphisms).

4. We introduce $\bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{1}{2}\gamma m_{\alpha\beta}$. Write the linearized Einstein equations in term of $\bar{\gamma}_{\alpha\beta}$.

5. Show that the linearization of the Einstein tensor is invariant by the transformation $\bar{\gamma}_{\alpha\beta} \to \bar{\gamma}_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \partial^\mu \xi_\mu m_{\alpha\beta}$ for all vector field ξ.

6. Show that one can choose ξ such that $\partial^\beta \bar{\gamma}_{\alpha\beta} = 0$.

7. How do you write the linearized Einstein equations in this gauge ?

Chapter 4

The wave equation

4.1 The Wave equation on Minkowski space-time : solution by spherical means

In this section, we will consider the wave equation

$$\partial_t^2 u - \Delta u = f,$$

subject to initial conditions $(u, \partial_t u)|_{t=0} = (g, h)$ where g and h are smooth functions on $\mathbb{R}^n$. The unknown is a function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$. On $\mathbb{R}^n$, there is an elegant way to solve the wave equation, which is the method of spherical means.

4.1.1 Solution for n=1

We can solve the one dimensional homogeneous wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \text{ in } \mathbb{R} \times [0, \infty) \\ u = g, \; \partial_t u = h \text{ on } \mathbb{R} \times \{0\} \end{cases}$$

We write

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0.$$

The solution of this equation can be written

$$u(x,t) = a(x+t) + b(x-t)$$

where the function a and b should satisfy

$$\begin{aligned} a(x) + b(x) &= g(x) \\ a'(x) - b'(x) &= h(x) \end{aligned}$$

Therefore we obtain

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi) d\xi. \tag{4.1.1}$$

4.1.2 Solution in 3 dimensions

We start with computations in $\mathbb{R}^n$ for any n. To solve the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{4.1.2}$$

we introduce the spherical means

$$M_u(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} u(y) dS_y = \frac{1}{\omega_n} \int_{|\xi|=1} u(x+r\xi) dS_\xi.$$

where ω_n is the area of the sphere of radius 1 in $\mathbb{R}^n$, and dS_y, and dS_ξ are the volume form on the sphere of radius r and of radius 1. Let us note that $M_u(x,0) = u(x)$. We calculate

$$\begin{aligned} \partial_r(M_u(x,r)) =& \frac{1}{\omega_n} \int_{|\xi|=1} \xi.\nabla u(x+r\xi) dS_\xi \\ =& \frac{1}{\omega_n r^{n-1}} \int_{|\eta|=r} \frac{\eta}{|\eta|}.\nabla u(x+\eta) dS_\eta \\ =& \frac{r}{\omega_n} \int_{|\xi|<1} \Delta_x u(x+r\xi) d\xi \\ =& \frac{r^{1-n}}{\omega_n} \Delta_x \int_{|y|<r} u(x+y) dy \\ =& r^{1-n} \Delta_x \int_0^r \rho^{n-1} M_u(x,\rho) d\rho. \end{aligned}$$

Consequently

$$\partial_r \left(r^{n-1} \partial_r M_u(x,r) \right) = \Delta_x r^{n-1} M_u(x,r)$$

so

$$\left(\partial_r^2 + \frac{n-1}{r} \partial_r \right) M_u = \Delta_x M_u.$$

If u satisfies $\Box u = 0$ then

$$\partial_t^2 M_u(x,r,t) = \left(\partial_r^2 + \frac{n-1}{r} \partial_r \right) M_u(x,r,t).$$

For $n = 3$ this yields

$$(\partial_t^2 - \partial_r^2)(r M_u) = 0.$$

Moreover for $t = 0$ and $r \geq 0$ we have $rM_u = rM_g$ and $\partial_t(rM_u) = rM_h$ and for $r = 0$ we have $rM_u = 0$. To solve the one dimensional wave equation on the half line $r \geq 0$, with the Dirichlet boundary condition at $r = 0$, we can extend first the initial data and the solution on the whole line as odd functions.

Exercise 1.

Show that a solution to the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{in } [0,\infty) \times \times [0,\infty) \\ u = g,\ u_t = h & \text{on } [0,\infty) \times \{0\} \\ u(0,t) = 0 & \text{for } t \geq 0 \end{cases}$$

can be written, for $x < t$,

$$u(x,t) = \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{t+x} h(\rho)d\rho.$$

We obtain, for $r < t$

$$rM_u(x,r,t) = \frac{1}{2}\left((r+t)M_g(x,r+t) - (t-r)M_h(x,t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \rho M_h(x,\rho)d\rho.$$

Consequently if we let $r \to 0$ we obtain

$$\begin{aligned} u(x,t) =& \partial_t(tM_g(x,t)) + tM_h(x,t) \\ =& \frac{1}{4\pi t^2}\int_{|x-y|=t}(th(y) + g(y) + \nabla g(y).(y-x))dS_y. \end{aligned}$$

This formula is called the Kirchoff's formula.

Remark 4.1.1. *This method allow also to obtain a formula in dimension $n = 2k+1$. Indeed if one set $U(r,t) = \left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}M_u\right)$ then U satisfies also the one dimensional transport equation.*

4.1.3 Solution in 2 dimensions

A solution to the wave equation (4.1.2) in 2 dimensions can be seen as a solution to the wave equation in 3 dimension, not depending on the third variable. We set

$$\tilde{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t), \quad \tilde{g}(x_1,x_2,x_3) = g(x_1,x_2), \quad \tilde{h}(x_1,x_2,x_3) = h(x_1,x_2).$$

We note also $x = (x_1,x_2)$ and $\tilde{x} = (x_1,x_2,x_3)$. We can compute

$$\frac{1}{4\pi t^2}\int_{|\tilde{y}-\tilde{x}|=t}\tilde{g}(\tilde{y})dS_{\tilde{y}} = \frac{1}{2\pi t}\int_{|y-x|\le t}\frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy.$$

Consequently the Kirchoff's formula for $\tilde{u}$ yield the following formula

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B(x,t)}\frac{t^2h(y) + tg(y) + t\nabla g(y).(y-x)}{\sqrt{t^2-|y-x|^2}}dy.$$

This formula is called the Poisson's formula, and the method used to obtain it the method of descent.

One can remark that there is a fundamental difference on how the solution is influenced by the initial data between 3 and 2 dimension (and actually between odd and even dimensions). In 3 dimension, the solution at (x,t) depends only on the initial data in an infinitesimal neighbourhood of the sphere of centre x and radius t. This implies for instance that if the initial data are supported in a compact set, after a long enough time, the solution will vanish in this compact. It is not any more the case in two dimensions ! As an example, one can compare the propagation of sound which obey a wave equation in 3 dimensions, to the ripples made by a stone launched on water, which obeys a wave equation in 2 dimensions.

4.1.4 Duhamel's principle

The solution of the inhomogeneous wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = 0,\ \partial_t u = 0 \text{ on } \mathbb{R}^n \times \{0\} \end{cases}$$

is given by $u(x,t) = \int_0^t U(x,t,s)ds$, where for all $s \geq 0$, U is the solution to

$$\begin{cases} \partial_t^2 U(x,t,s) - \Delta U(x,t,s) = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ U(x,s,s) = 0,\ \partial_t U(x,s,s) = f(x,s) \text{ on } \mathbb{R} \times \{s\} \end{cases}$$

This principle, which is not specific to the wave equation, is called the Duhamel's principle. It allows to obtain a representation formula for the solutions to the inhomogeneous wave equation thanks to Kirchoff or Poisson's formula.

There is a drawbacks to the formula obtained via spherical means : they seem to require a lot of regularity for the initial data. We will see an other method of resolution, which is more robust, meaning that it can be better adapted to perturbed problems, and which highlight the "hyperbolicity" of the wave equation, which is a property of propagation of the regularity.

4.2 The Wave equation on Minkowski space-time : the energy method

4.2.1 Conservation of energy

We start with a formal computation. Assume that u is a solution to

$$\begin{cases} \partial_t^2 u - \Delta u = F \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ \partial_t u = h \text{ on } \mathbb{R}^n \times \{0\} \end{cases} \tag{4.2.1}$$

If we multiply both sides of the equation by $\partial_t u$ we obtain

$$\begin{aligned} F\partial_t u =& \partial_t u \left(\partial_t^2 u - \Delta u\right) \\ =& \frac{1}{2}\partial_t(\partial_t u)^2 + div(\partial_t u \nabla u) - \nabla\partial_t u.\nabla u \\ =& \frac{1}{2}\partial_t\left((\partial_t u)^2 + |\nabla u|^2\right) + div(\partial_t u \nabla u) \end{aligned}$$

Let

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} ((\partial_t u)^2 + |\nabla u|^2)dx.$$

The above equality yields the so called conservation of energy

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{\mathbb{R}^n} 2F\partial_t u dx.$$

This equality shows that the more natural function spaces to study wave equations are Sobolev spaces : in these spaces the regularity of the solution corresponds to the regularity of the initial data.

4.2.2 The Sobolev space H^s

A very handy way to define the Sobolev spaces $H^s(\mathbb{R}^n)$ is via the Fourier transform. The Fourier transform $\hat{u}$ of a function $u \in L^1$ is defined by

$$\hat{u}(\xi) = \int e^{-ix.\xi} u(x)dx.$$

This definition can be extended "by duality" to tempered distributions (noted $\mathcal{S}'$).

Proposition 4.2.1.
- *The Fourier transform exchange derivation and multiplication :* $\widehat{\partial_j u}(\xi) = i\xi_j \hat{u}(\xi)$.
- *The Schwartz space $\mathcal{S}$ is stable by Fourier transform*
- *The Fourier transform is, up to a constant, a bijective isometry on $L^2(\mathbb{R}^n)$ whose inverse is given by*

$$u(x) = \frac{1}{(2\pi)^d} \int e^{ix.\xi} \hat{u}(\xi) d\xi.$$

More precisely, he have the Parseval equality

$$\frac{1}{(2\pi)^d} \int \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi = \int u(x)\overline{v(x)}dx.$$

Definition 4.2.2. *Let $s \in \mathbb{R}$. We say that a tempered distribution u belongs to the Sobolev space H^s if $\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi < \infty$. We set then*

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

We say that a tempered distribution u belongs to the homogeneous Sobolev space $\dot{H}^s$ if $\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi < \infty$. We set then

$$\|u\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Proposition 4.2.3. *For all $s \in \mathbb{R}$, H^s, equipped with the norm $\|.\|_{H^s}$ is a Hilbert space.*

Proposition 4.2.4. *If $m \in \mathbb{N}$, H^m is exactly the vector space of function $u \in L^2$ whose derivatives of order less or equal to m are also in L^2. Moreover*

$$\tilde{\|}u\|_{H^m} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}^2 \right)$$

is an Hilbert norm on H^m, equivalent to $\|.\|_{H^m}$.

Proposition 4.2.5. *Let $s \in \mathbb{R}$*

- *The space $\mathcal{D}(\mathbb{R}^d)$ of C^∞ compactly supported functions is dense in $H^s(\mathbb{R}^d)$.*

- *For all $s < t$, $H^t \subset H^s$ and we have the inequality*

$$\forall \theta \in [0,1], \|u\|_{H^{\theta s+(1-\theta)t}} \leq \|u\|_{H^s}^{\theta} \|u\|_{H^t}^{1-\theta}.$$

- *The multiplication by a function of $\mathcal{S}$ is a continuous function from H^s to itself.*

To characterise the elements of $(H^s)'$ we will use the following proposition

Proposition 4.2.6. *The map $f \mapsto (2\pi)^d < f, . >_{H^{-s} \times H^s}$ is an isometric isomorphism from H^{-s} to $(H^s)'$.*

Proof. Let $f \in H^{-s}$. The linear form

$$\phi_f : u \in \mathcal{S} \mapsto \frac{1}{(2\pi)^d} \int \hat{f} \overline{\hat{\hat{u}}} d\xi = \langle f, u \rangle_{\mathcal{S}', \mathcal{S}},$$

satisfies, thanks to Cauchy-Schwartz inequality

$$\phi(f)(u) \leq \frac{1}{(2\pi)^d} \|f\|_{H^{-s}} \|u\|_{H^s}.$$

Therefore, it can be extended to a linear form on H^s.

Conversely, if $u \in (H^s)'$, we can consider the linear form $\tilde{u}$ on $(L^2)'$, defined by

$$\tilde{u}(f) = u(\Lambda^{-s} f),$$

where

$$\begin{aligned} \Lambda^s : H^t &\to H^{t-s} \\ f &\mapsto \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{s}{2}} \hat{f}), \end{aligned}$$

and use Riesz representation theorem on L^2. □

We will need in the next chapter the following Sobolev embeddings

Proposition 4.2.7. *Let $s \geq 0$.*

- *If $s > \frac{d}{2}$ then $H^s(mR^d)$ is an algebra which continuously embeds in $C_0(\mathbb{R}^d)$ the space of continuous functions which tend to zero at infinity.*

- *If $0 \leq s < \frac{d}{2}$, let p_c be the critical exponent defined by $-s + \frac{d}{2} = \frac{d}{p_c}$, i.e $p_c = \frac{2d}{d-2s} \in [2, \infty[$. Then for all $p \in [2, p_c]$, $H^s(\mathbb{R}^d)$ continuously embeds in $L^p(\mathbb{R}^d)$:*

$$\exists C_{p,s} > 0 \text{ such that } \forall f \in H^s(\mathbb{R}^d), \ \|f\|_{L^p} \leq C_{p,s} \|f\|_{H^s}.$$

- *For $s = \frac{d}{2}$, $H^s(\mathbb{R}^d)$ continuously embeds in all the $L^p(\mathbb{R}^d)$ for all $2 \leq p < \infty$.*

4.2.3 Propagation of H^s norms

We note $\|\partial u\|_{H^s} = \|\partial_t u\|_{H^s} + \sum_{i=1}^n \|\partial_i u\|_{H^s}$.

Theorem 4.2.8. *Let $s \in \mathbb{R}$. If $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ and $\Box u \in C([0,T], H^s)$ then for all $0 < t < T$*

$$\|\partial u(t)\|_{H^s} \leq C \left(\|\partial u(0)\|_{H^s} + \int_0^t \|\Box u(\tau,.)\|_{H^s} d\tau \right).$$

Proof. Let us prove first the inequality in the case $s = 0$. In this case, the equality

$$(\Box u)\partial_t u = \frac{1}{2}\partial_t \left((\partial_t u)^2 + |\nabla u|^2 \right) + div(\partial_t u \nabla u),$$

is satisfied in the sense of distributions in $[0,t] \times \mathbb{R}^n$. We can consider for instance a cut-off function of the form $\chi(\varepsilon x)\phi(t)$. We have

$$\int_{[0,T]\times\mathbb{R}^n} \chi(\varepsilon x)\phi(t)(\Box u)\partial_t u = -\frac{1}{2}\int_{[0,T]\times\mathbb{R}^n} \left(\chi(\varepsilon x)\phi'(t) \left((\partial_t u)^2 + |\nabla u|^2 \right) + \varepsilon\phi(t)\partial_t u \nabla u.\nabla\chi(\varepsilon x) \right).$$

Consequently, letting $\varepsilon \to 0$, and using Fubbini's theorem, we obtain

$$\int_0^T \phi(t) \int_{\mathbb{R}^n} (\Box u)\partial_t u dx dt = -\frac{1}{2}\int_0^t \phi'(t) \int_{\mathbb{R}^n} \left((\partial_t u)^2 + |\nabla u|^2 \right) dx dt,$$

which means that in the sense of distributions

$$\frac{d}{dt}\mathcal{E}(t) = \int_{\mathbb{R}^2} (\Box u)\partial_t u dx \leq \|\Box u\|_{L^2} \sqrt{\mathcal{E}(t)},$$

so $\frac{d}{dt}\sqrt{\mathcal{E}(t)} \leq \|\Box u\|_{L^2}$, which implies

$$\|\partial u(t)\|_{L^2} \leq C(\|\partial u(0)\|_{L^2} + \int_0^t \|\Box u(\tau,.)\|_{L^2} d\tau).$$

In the case $s \neq 0$, we use the operator Λ^s and notice that $\Box\Lambda^s u = \Lambda^s \Box u$: we can apply the case $s = 0$ to $\Lambda^s u$ to obtain the desired result.

$\Box$

4.2.4 Existence and uniqueness of solutions

We can now state a theorem about existence and uniqueness os solutions to (4.2.1).

Theorem 4.2.9. *Let $s \in \mathbb{Z}$ and $f \in H^{s+1}(\mathbb{R}^n), h \in H^s(\mathbb{R}^n), F \in C([0,T], H^s)$. There exists a unique solution to* (4.2.1) *in $C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$.*

Proof. We first prove the unicity : if u and v are functions in $C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ which satisfy (4.1.2) then $u - v$ is in $C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ and satisfy $\Box(u - v) = 0$, $((u - v), \partial_t(u - v))|_{t=0} = (0, 0)$. Consequently Theorem 4.2.8 yields for all t

$$\|\partial(u - v)(t)\|_{H^s} = 0,$$

which implies $u = v$. We now prove the existence. If the initial data are regular enough, we can obtain a solution with the method of spherical means. A more suited way would be to use the Fourier transform to obtain an other representation formula. Formally, taking the space Fourier transform of the equation (4.2.1) we obtain

$$\partial_t^2 \hat{u} - |\xi|^2 \hat{u} = \hat{F}$$

Solving this differential equation at fixed ξ with initial data $\hat{u}(0,\xi) = \hat{f}(\xi),\ \partial_t \hat{u}(0,\xi) = \hat{h}(\xi)$ we obtain

$$\hat{u}(\xi) = \frac{\hat{h}(\xi)}{2\pi|\xi|}\sin(2\pi t\xi) + \hat{f}(\xi)\cos(2\pi t\xi) + \int_0^t \frac{\hat{F}(s,\xi)}{2\pi|\xi|}\sin(2\pi(t-s)\xi)ds.$$

We can check a posteriori that the inverse transform of the above formula is a solution to (4.2.1) which is in $C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$.

□

4.2.5 Finite speed of propagation

Theorem 4.2.10. *Let $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ be a solution of (4.2.1) such that for some $x_0, t_0 < T$ we have $(f,h)|_{B(x_0,t_0)} = (0,0)$ and $F|_{K(x_0,t_0)} = 0$ where $K(x_0,t_0) = \{(x,t) \in \mathbb{R}^n \times [0,t_0]; |x-x_0| \leq t_0 - t\}$. Then u is zero in $K(x_0,t_0)$.*

Proof. We can apply Stokes theorem to

$$f\partial_t u = \frac{1}{2}\partial_t\left((\partial_t u)^2 + |\nabla u|^2\right) + div(\partial_t u \nabla u)$$

in the domain $K(x_0,t_0) \cup [0,t]$. □

4.3 The wave equation with variable coefficients

In this section, we will look at solutions of

$$\begin{cases} \Box_g u = F \ \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = f,\ \partial_t u = h \ \text{ on } \mathbb{R}^n \times \{0\} \end{cases} \tag{4.3.1}$$

where g is a Lorentzian metric. We recall that

$$\Box_g u = D^\alpha \partial_\alpha u = \frac{1}{\sqrt{|det(g)|}}\partial_\alpha(g^{\alpha\beta}\sqrt{|det(g)|}\partial_\beta u).$$

4.3.1 Energy identities

There is a geometric way of seeing the energy identities for solutions to the wave equation $\Box_g u = F$, where g is a Lorentzian metric on a maniflod $\mathcal{M}$. The stress energy-tensor of the scalar field u is given by

$$Q_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu}\partial_\mu u \partial_\nu u.$$

We can compute the divergence of Q :

$$\begin{aligned} D^\alpha Q_{\alpha\beta} =& \partial_\beta u D^\alpha \partial_\alpha u + \partial_\alpha u D^\alpha \partial_\beta u - g_{\alpha\beta} g^{\mu\nu} \partial_\mu u D^\alpha \partial_\nu u \\ =& \partial_\beta u \Box_g u + \partial_\alpha u D^\alpha \partial_\beta u - g^{\mu\nu} \partial_\mu u D_\beta \partial_\nu u \\ =& \partial_\beta u \Box_g u + \partial_\alpha u D^\alpha \partial_\beta u - g^{\mu\nu} \partial_\mu u D_\nu \partial_\beta u \\ =& \partial_\beta u \Box_g u \end{aligned}$$

where we used that $D_\nu \partial_\beta u = D_\beta \partial_\nu u$. Consequently, if we contract with a vector field T we obtain

$$T(u)\Box_g u = T^\beta \partial_\beta u \Box_g u = T^\beta D^\alpha Q_{\alpha\beta} = D^\alpha (T^\beta Q_{\alpha\beta}) - Q_{\alpha\beta} D^\alpha T^\beta = D^\alpha (T^\beta Q_{\alpha\beta}) - \frac{1}{2} Q_{\alpha\beta}{}^T\pi^{\alpha\beta},$$

where ${}^T\pi$ is the deformation tensor of T, which we recall is zero if T is a Killing vector field.

Let us assume that $\mathcal{M}$ can be foliated by space-like hypersurfaces Σ_t, indexed by a time function t. Let T be the vector field of unit normal to Σ_t, and let apply Stokes theorem in the region $\cup_{s\in[0,t]}\Sigma_s$. We have

$$\int_{\Sigma_t} T^\alpha T^\beta Q_{\alpha\beta} dvol_{\bar g} = \int_{\Sigma_0} T^\alpha T^\beta Q_{\alpha\beta} dvol_{\bar g} + \int_{\cup_{s\in[0,t]}\Sigma_s} \left(\frac{1}{2} Q_{\alpha\beta}{}^T\pi^{\alpha\beta} + FT(u) \right) dvol_g.$$

Proposition 4.3.1. *We will assume, to simplify the geometric considerations, that we have a global coordinate chart (t, x^i) in which*

$$|g_{\alpha\beta} - m_{\alpha\beta}| \leq \frac{1}{8}. \tag{4.3.2}$$

We assume also that $\partial_t g \in L^1([0,T], L^\infty(\mathbb{R}^n))$, $F \in L^1([0,T], L^2(\mathbb{R}^n))$, $(f,h) \in \dot H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Let $u \in C([0,T], H^1) \cap C^1([0,T], L^2)$ be a solution of (4.3.1). We have

$$\|u, \partial_t u\|_{\dot H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \lesssim \left(\|f,h\|_{H^1(\dot{\mathbb{R}}^n)\times L^2(\mathbb{R}^n)} + \int_0^t \|F\|_{L^2(\mathbb{R}^n)}(s) ds \right) \exp(\int_0^t \|\partial_t g\|_{L^\infty(\mathbb{R}^n)}(s) ds).$$

Proof. Let $T = \partial_t$. Then we have $T^\alpha T^\beta Q_{\alpha\beta} = (\partial_t\phi)^2 - \frac{1}{2} g_{00} \left(g^{00}(\partial_t\phi)^2 + g^{ij}\partial_i\phi\partial_j\phi + 2g^{0i}\partial_t\phi\partial_i\phi \right)$. Under the hypothesis (4.3.2) their exist a constant C such that

$$C((\partial_t u)^2 + |\nabla u|^2) \leq Q_{00} \leq \frac{1}{C}((\partial_t u)^2 + |\nabla u|^2).$$

We also calculate

$${}^{\partial_t}\pi_{\alpha\beta} = (\mathcal{L}_{\partial_t} g)_{\alpha\beta} = \partial_t g_{\alpha\beta}.$$

Consequently we have the estimate

$$|Q_{\alpha\beta}{}^{\partial_t}\pi_{\alpha\beta}| \leq C((\partial_t u)^2 + |\nabla u|)^2 |\partial_t g_{\alpha\beta}.$$

We have calculated

$$\int_{\Sigma_t} Q_{00} dvol_{\bar g} = \int_{\Sigma_0} Q_{00} dvol_{\bar g} + \int_{\cup_{s\in[0,t]}\Sigma_s} \left(\frac{1}{2} Q_{\alpha\beta}{}^{\partial_t}\pi^{\alpha\beta} + F\partial_t u \right) dvol_g.$$

We will note $|\partial u|^2 = (\partial_t u)^2 + |\nabla u|^2$. We have

$$\int_{\Sigma_t} (\partial u)^2 dx \leq C' \int_0^t \left(\|\partial_t g\|_{L^\infty(\Sigma_s)} \int_{\Sigma_s} (\partial u)^2 dx + \|F\|_{L^2(\Sigma_s)} \left(\int_{\Sigma_s} (\partial u)^2 dx \right)^{\frac{1}{2}} \right) ds + \int_{\Sigma_0} (\partial u)^2 dx. \tag{4.3.3}$$

Using the inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ we write

$$\sup_{s\in[0,t]} \int_{\Sigma_s} (\partial u)^2 dx \leq C' \int_0^t \|\partial_t g\|_{L^\infty} \left(\int_{\Sigma_s} (\partial u)^2 dx \right) ds + \frac{1}{\varepsilon} \left(\int_0^t \|F\|_{L^2(\Sigma_s)} ds \right)^2$$
$$+ \varepsilon \sup_{s\in[0,t]} \int_{\Sigma_s} (\partial u)^2 dx + \int_{\Sigma_0} (\partial u)^2 dx.$$

The term with a factor ε can be absorbed by the left hand-side. We conclude with Gronwall lemma that we recall in the next paragraph. $\square$

We recall here Gronwall lemma

Lemma 4.3.2. *Let $\phi, \psi : [0,T] \to \mathbb{R}^+$ such that ϕ is continuous and ψ is integrable. We assume that their exists $A \geq 0$ such that for all $t \in [0,T]$*

$$f(t) \leq A + \int_0^t f(s) g(s) ds.$$

Then we have

$$f(t) \leq A \exp\Big(\int_0^t g(s) ds \Big).$$

Corollary 4.3.3. *Let $m \in \mathbb{N}$. In addition to the hypothesis of Proposition 4.3.1 we assume $\partial g \in L^1([0,T], C^m(\mathbb{R}^n))$, $F \in L^1([0,T], H^m(\mathbb{R}^n))$, $(f,h) \in H^{m+1}(\mathbb{R}^n) \times H^m(\mathbb{R}^n)$. Let $u \in C([0,T], H^{m+1}) \cap C^1([0,T], H^m)$ be a solution of (4.3.1). We have*

$$\|\partial u\|_{H^m(\mathbb{R}^n)}(t) \lesssim \left(\|\nabla f, h\|_{H^m(\mathbb{R}^n)} + \int_0^t \|F\|_{H^m(\mathbb{R}^n)}(s) ds \right) \exp\Big(\int_0^t \|\partial g\|_{C^m(\mathbb{R}^n)}(s) ds \Big).$$

Moreover

$$\|u\|_{H^m} \leq (1+T) \left(\|f,h\|_{H^{m+1}(\mathbb{R}^n)\times H^m(\mathbb{R}^n)} + \int_0^t \|F\|_{H^m(\mathbb{R}^n)}(s) ds \right) \exp\Big(\int_0^t \|\partial g\|_{C^m(\mathbb{R}^n)}(s) ds \Big).$$

Proof. We write the equation for $\nabla^m u$, any space derivative of order m : $\Box_g \nabla^m u = [\Box_g, \nabla^m] u + \nabla^m F$ and we apply Proposition 4.3.1. For this purpose, we have to estimate $[\Box_g, \nabla^m] u$ in L^2. We can write

$$|[\Box_g, \nabla^m] u| \lesssim \sum_{i_0+i_1+\ldots+i_m \leq m+1,\ i_0 \leq m} |\partial^{i_0} \partial u| |\partial^{i_1} g| .. |\partial^{i_m} g|,$$

And consequently

$$\|[\Box_g, \nabla^m] u\|_{L^2} \leq C(\|g\|_{C^{m+1}}) \|\partial u\|_{H^m}.$$

Consequently (4.3.3) written for $\nabla^m u$ yields

$$\|\partial u\|^2_{H^m(\Sigma_t)} \leq C' \left(\|\partial u\|^2_{H^m(\Sigma_0)} + \int_0^t \left(C(\|g\|_{C^{m+1}}) \|\partial u\|^2_{H^m} + \|\nabla^m F\|_{L^2} \|\partial u\|_{H^m} \right) ds \right)$$

Once again we can conclude with Gronwall.

The estimate for $\|u\|_{L^2}$ which is what is added in the second estimate of Corollary 4.3.3 is a consequence of

$$\|u\|_{L^2(\Sigma_t)} \leq \|u\|_{L^2(\Sigma_0)} + \int_0^t \|\partial_t u\|_{L^2} \leq \|u\|_{L^2(\Sigma_0)} + t \sup_{[0,t]} \|\partial_t u\|_{L^2}.$$

$\square$

4.3.2 Existence of solution to (4.3.1)

The energy identities of Proposition 4.3.1 and Corollary 4.3.3 show directly the unicity of the solutions to (4.3.1). In this section we will see how, by duality, these a priori energy estimate can give us the existence.

Theorem 4.3.4. *Let $m \in \mathbb{N}$. In addition to* (4.3.2) *we assume $\partial g \in L^1([0,T], C^{m+2}(\mathbb{R}^n))$, $F \in L^1([0,T], H^m(\mathbb{R}^n))$, $(f,h) \in H^{m+1}(\mathbb{R}^n) \times H^m(\mathbb{R}^n)$. Their exists a unique solution*

$$u \in C([0,T], H^{m+1}) \cap C^1([0,T], H^m),$$

of (4.3.1).

The proof will proceed by duality, using Hahn Banach's theorem that we recall here

Theorem 4.3.5. *Let G be a subvector space of E, a normed vector space, and $g : G \to \mathbb{R}$ be a linear continuous form of norm $\|g\|_{G'} = \sup_{x \in G, \|x\| \leq 1} g(x)$. Then there exists a continuous linear form $f \in E'$ such that $f|G = g$ and $\|f\|_{E'} = \|g\|_{G'}$.*

You may find the proof in [1] We will need also a-priori estimates for solutions of (4.3.1) in H^{-m}.

Lemma 4.3.6. *Let $m \in \mathbb{N}$. In addition to the hypothesis of Proposition 4.3.1 we assume $\partial g \in L^1([0,T], C^{m+2}(\mathbb{R}^n))$, $F \in L^1([0,T], H^{-m}(\mathbb{R}^n))$, $(f,h) \in H^{-m+1}(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n)$. Let*

$$u \in C([0,T], H^{-m+1}) \cap C^1([0,T], H^{-m}),$$

be a solution of (4.3.1). *We have*

$$\|\partial u\|_{H^{-m}(\mathbb{R}^n)}(t) \lesssim \left(\|\nabla f, h\|_{H^{-m}(\mathbb{R}^n)} + \int_0^t \|F\|_{H^{-m}(\mathbb{R}^n)}(s) ds \right) \exp(\int_0^t \|\partial g\|_{C^{m+2}(\mathbb{R}^n)}(s) ds). \tag{4.3.4}$$

Proof. We proceed by induction on m : we know from Corollary 4.3.3 that the property is true for $m = -1, 0$. We will now prove that if (4.3.4) holds for some m_0, it holds for $m_0 + 2$: Let $u \in C([0,T], H^{-m_0-1}) \cap C^1([0,T], H^{-m_0-2})$. Let

$$v = \Lambda^{-2} u = (1 - \Delta)^{-1} u \in C([0,T], H^{-m_0+1}) \cap C^1([0,T], H^{-m_0})$$

We have

$$
\begin{aligned}
&\|\partial u\|_{H^{-m_0-2}(\mathbb{R}^n)} \\
=&\|\partial v\|_{H^{-m_0}} \\
\leq &\left(\|\Lambda^{-2}\nabla f, \Lambda^{-2}h\|_{H^{-m_0}(\mathbb{R}^n)} + \int_0^t \|\Box_g v\|_{H^{-m_0}(\mathbb{R}^n)}(s)ds\right)\exp(\int_0^t \|\partial g\|_{C^{m_0}(\mathbb{R}^n)}(s)ds).
\end{aligned}
$$

We can write

$$\Box_g u = (1-\Delta)\Box_g v + [1-\Delta, \Box_g]v.$$

Moreover, $[1-\Delta, \Box_g]v$ is a sum of products of up to 2 derivatives of ∂g and up to 2 derivatives of ∂v. We can use the result of Exercise (4.4) which says that

$$\|ab\|_{H^{-k}} \leq C\|a\|_{C^k}\|b\|_{H^k}$$

to write

$$
\begin{aligned}
\|\Box_g v\|_{H^{-m_0}} =&\|(1-\Delta)\Box_g v\|_{H^{-m_0+2}} \\
\leq& \|\Box_g u\|_{H^{-m_0+2}} + \|[1-\Delta, \Box_g]v\|_{H^{-(m_0+2)}} \\
\leq& \|\Box_g u\|_{H^{-m_0+2}} + C(\|\partial g\|_{C^{m_0+4}})\|\partial v\|_{H^{-m_0}}
\end{aligned}
$$

so once again we can conclude with Gronwall lemma. □

We are now ready to prove Theorem 4.3.4.

Proof. We consider first the case where $f = h = 0$. Let $E = L^1([-1,T], H^{-m-1})$ and let G be the subset of E which consists of functions w such that their exists $v \in C_c^\infty((-1,T), \mathbb{R}^n)$ such that $w = \Box_g v$ in $[0,T]\times\mathbb{R}^n$. Let $\phi : G \to \mathbb{R}$ defined by

$$\phi(w) = \int_0^T \int_{\mathbb{R}^n} vF dvol_g,$$

We have

$$
\begin{aligned}
|\phi(w)| \leq& \int_0^T \|F\|_{H^m(\mathbb{R}^n)}\|v\|_{H^{-m}}dt \\
\leq& C(T,g)\left(\int_0^T \|F\|_{H^m(\mathbb{R}^n)}dt\right)\left(\int_0^T \|\Box_g v\|_{H^{-m-1}(\mathbb{R}^n)}\right) \\
\leq& C(T,g)\left(\int_0^T \|F\|_{H^m(\mathbb{R}^n)}dt\right)\|w\|_{L^1([-1,T],H^{-m-1}(\mathbb{R}^n))}.
\end{aligned}
$$

where we have used the energy estimate on $C([0,T], H^{-m}) \cap C^1([0,T], H^{-m-1})$ for the backward solution v to the equation $\Box_g v = w$ with data $(v, \partial_t v)|_{t=T} = (0,0)$.

Consequently, with Hahn-Banach theorem, we know that there exists $\Phi \in L^1([-1,T], H^{-m-1})'$, which coincides with ϕ on G and such that $\|\Phi\| \leq C(T,g)(\int_0^T \|F\|_{H^m}dt)$. Φ can be represented by a function $u \in L^\infty([-1,T], H^{m+1})$ which satisfies, for all $v \in C_c^\infty((0,T)\times\mathbb{R}^n)$

$$\int_0^T \int_{\mathbb{R}^n} vF dvol_g = \int_0^T \int_{\mathbb{R}^n} u\Box_g v dvol_g = \int_0^T \int_{\mathbb{R}^n} (\Box_g u)v dvol_g,$$

which yields $\Box_g u = F$ on $[0,,T]\times\mathbb{R}^n$. Moreover, for all $h \in C_c^\infty((-1,0)\times\mathbb{R}^n)$ we have $\int uhdvol_g = 0$, which yields $u(t,x) = 0$ for $t < 0$. We can write, for all $v \in C_c^\infty((-1,T)\times\mathbb{R}^n)$

$$\int_0^T \int_{\mathbb{R}^n} Fvdvol_g = \int_{-1}^T \int_{\mathbb{R}^n} (\Box_g u)vdvol_g = \int_{-1}^T \int_{\mathbb{R}^n} g^{\alpha\beta}\partial_\alpha u \partial_\beta v dvol_g.$$

This yields

$$\int_{-1}^T \int_{\mathbb{R}^N} g^{NN} N(v)N(u) \leq \quad C(\|F\|_{L^1([0,T],H^m)}\|v\|_{L^\infty([0,T],H^{-m})} + \|\nabla u\|_{L^\infty([0,T]H^m)}\|\nabla v\|_{L^1([0,T],H^{-m})})$$
$$\leq C\|N(v)\|_{L^1([0,T],H^{-m+1})},,$$

where we have noted N the normal to Σ_t. This yields $\partial_t u \in L^\infty([0,T],H^{m-1})$, and so $u \in C([0,T],H^{m-1})$. Thanks to the equation $\Box_g u = F$ we have also $\partial_t^2 u \in L^\infty([0,T],H^{m-2})$, and so $u \in C^1([0,T],H^{m-2})$. This allows us to conclude that $(u,\partial_t u)|_{t=0} = (0,0)$

Let us now consider the case $F = 0$ but f and h non zero. Let $v(x,t) = f(x) + th(x)$. We have $\Box_g v \in L^1([0,T],H^{m-1})$. Thanks to what we have done above, we know that their exists $\Box_g w = -\Box_g v$, and $(w,\partial_t w)|_{t=0} = (0,0)$. Let now $u = w + v$: it is a solution to the equation.

For the moment, we have shown existence of solutions with an apparent loss of regularity. To prove the propagation of the desired regularity, we can approximate F, f, h and the metric g by smooth function and metrics F_n, f_n, h_n and g_n. For all n, what we have done above allows us to find $u_n \in C^1([0,T],H^k)$ for any k, solution of

$$\begin{cases} \Box_{g_n} u_n = F_n & \text{in } \mathbb{R}^n \times (0,\infty) \\ u_n = f_n,\ \partial_t u_n = h_n & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

Then Corollary 4.3.3 allows to show that u_n is a Cauchy sequence in $C^1([0,T],H^m)\cap C^0([0,T],H^{m+1})$. The limit is our solution u and has the desired regularity. $\Box$

4.4 Exercises

Exercise 2.

The aim of this equation is to derive the conservation laws for the wave equation. Let (M,g) be a Lorentzian manifold. We consider a solution to the wave equation $\Box_g u = 0$. We recall that $\Box_g u = D^\alpha \partial_\alpha u$.

1. Let $T_{\mu\nu} = \partial_\mu u \partial_\nu u - \frac{1}{2} g_{\mu\nu}\partial^\alpha u \partial_\alpha u$. Show that $D^\mu T_{\mu\nu} = 0$.

2. Let K be a Killing field for the metric g. Show that $D^\mu(T_{\mu\nu}K^\nu) = 0$.

3. We consider now the case $(M,g) = (\mathbb{R}^{1+3}, m)$, and $\Box u = 0$. Show that the above formula, with $K = \partial_t$ yields the energy identity when integrating over a space-time slab. What do you obtain with K the other Killing fields of Minkowski metric ?

Exercise 3.

The aim of this exercise is to show that H^s is an algebra for $s > \frac{n}{2}$.

1. Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. Show that $\mathcal{F}(u * v) = \mathcal{F}(u)\mathcal{F}(v)$. Where we have noted $\mathcal{F}$ the Fourier transform, and $*$ the convolution

$$u * v(x) = \int_{\mathbb{R}^n} u(y)v(x-y).$$

Deduce $\mathcal{F}(uv)$.

2. We want to show that $H^s(\mathbb{R}^n)$ is an algebra for $s > \frac{n}{2}$. Let $u, v \in \mathcal{S}$: express the H^s norm of uv.

3. Show that for $s > 0$ we have

$$(1+|\xi|^2)^{\frac{s}{2}} \leq C_s((1+|\xi-\eta|^2)^{\frac{s}{2}} + (1+|\eta|^2)^{\frac{s}{2}}).$$

4. Use Minkowski inequality

$$\left(\int\left(\int f(x,y)dx\right)^2 dy\right)^{\frac{1}{2}} \leq \int\left(\int f(x,y)^2 dy\right)^{\frac{1}{2}} dx$$

to prove that, for $s > 0$

$$\|uv\|_{H^s} \leq C_s(\|\mathcal{F}u\|_{L^1}\|v\|_{H^S} + \|\mathcal{F}v\|_{L^1}\|u\|_{H^S}).$$

5. Conclude that H^s is an algebra for $s > \frac{n}{2}$.

Exercise 4.

Let u be a solution to the wave equation $\Box u = 0$ with smooth compactly supported initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$.

1. Show that $\Box Z u = 0$ for $Z \in \{\partial_\alpha, \Omega_{\alpha,\beta}, S\}$ where $\Omega_{ij} = x^i\partial_j - x^j\partial_i$, $\Omega_{0i} = t\partial_i + x^i\partial_t$, $S = t\partial_t + r\partial_r$, with r the polar coordinate $r = |x|$.

2. Show that $\|\partial Z u(t)\|_{L^2} \leq C(u_0, u_1)$.

3. We have the following inequality, called Klainerman-Sobolev inequality for functions in $\mathbb{R}^n$:

$$(1+t+r)^{\frac{n-1}{2}}(1+|r-t|)^{\frac{1}{2}}|f(t,x)| \leq C_n \sum_{|I|\leq\frac{n+1}{2}} \|Zf\|_{L^2(\mathbb{R}^n)}.$$

Deduce a decay estimate for the solution u to the wave equation.

Exercise 5.

1. Let $u \in H^m(\mathbb{R}^n)$ with $m \in \mathbb{N}$. Show that for all $v \in C_c^\infty$ we have

$$\|uv\|_{H^m} \leq \|v\|_{C^m}\|u\|_{H^m}.$$

2. We now assume $u \in H^{-m}$ with $m \in \mathbb{N}$. Show that

$$\|uv\|_{H^{-m}} \leq \|v\|_{C^m}\|u\|_{H^{-m}}.$$

Exercise 6.

We recall that the metric in the exterior of a Schwarzschild black-hole can be written

$$g = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

1. Express $\Box_g \phi$ in the coordinates (t, r, θ, ϕ).

2. $\frac{\partial}{\partial t}$ is a Killing field : what is the conserved energy associated to it ?

3. We consider a solution to $\Box_g \phi = F$, with initial data $(\phi, \partial_t \phi)|_{t=0} = (u_0, u_1)$: what estimate do you obtain on $\partial \phi$?

4. Recall the expression of g in the coordinates t, r^*, θ, ϕ where $r^* = r + 2m \ln(r - 2m)$. Show that if ϕ satisfies $\Box_g \phi = 0$ and $(\phi, \partial_t \phi)|_{t=0}$ is zero for $a \leq r^* \leq b$ then $\phi(t, x)$ is zero for $b + t \leq r^* \leq a - t$.

Exercise 7.

The aim of this exercise is to prove Klainerman Sobolev inequality.

1. Show that for all function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ we have

$$|\partial_\alpha f(t,x)| \leq \frac{1}{1 + |t - x|} \sum_{Z \in \mathcal{Z}} |Zf|$$

where $\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha,\beta}, S\}$.

2. Let $f \in C_c^\infty(\mathbb{R}^{n+1})$. We write

$$f = f_1 + f_2,$$

where

$$f_1 = \chi\left(\frac{r}{t}\right) f, \quad f_2 = \left(1 - \chi\left(\frac{r}{t}\right)\right) f,$$

and χ is a cut-off such that $\chi(\rho) = 1$ for $\rho \leq \frac{1}{2}$ and $\chi(\rho) = 0$ for $\rho \geq \frac{2}{3}$. By applying the Sobolev embeding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $s > \frac{n}{2}$ to $f_t = f_1(t, tx)$. show that

$$|f_1(t,x)| \lesssim \frac{1}{t^{\frac{n}{2}}} \sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|t^\alpha \nabla^\alpha f_1\|_{L^2(\mathbb{R}^n)}.$$

3. With the first question, deduce that

$$t^{\frac{n}{2}} |f_1(t,x)| \lesssim \sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|Z^\alpha f_1\|_{L^2(\mathbb{R}^n)}.$$

4. We consider f_2. Using the following inequality (where we note by θ the coordinates on the sphere),

$$(1 + t + r)(1 + |t - r|)(f_2(t, r, \theta))^2 \lesssim \int_{\frac{t}{2}}^{r} \partial_\rho \left((1 + t + \rho)(1 + |t - \rho|) f_2(t, \rho, \theta)^2\right) d\rho,$$

the standard Sobolev inequality on the sphere and the first question, show that f_2 satisfy the Klainerman Sobolev inequality

$$(1 + t + r)^{\frac{n-1}{2}} (1 + |r - t|)^{\frac{1}{2}} |f(t,x)| \leq C_n \sum_{|I| \leq \frac{n}{2} + 1} \|Z^I f\|_{L^2(\mathbb{R}^n)}.$$

5. Conclude.

Chapter 5

A local existence result

5.1 Local well-posedness for a semi-linear wave equation

In this short chapter, we want to show local existence of solutions for a non linear equation of the form

$$\begin{cases} \Box u = (\partial_t u)^2, & \text{in } \mathbb{R}^d \times \mathbb{R} \\ (u, \partial_t u) = (u_0, u_1) & \text{in } \mathbb{R}^d \times \{0\} \end{cases} \tag{5.1.1}$$

This equation is a toy model for Einstein equations. Our analysis can easily be generalized to equations of the form $\Box u = F(x, u, \partial u)$ where F is a smooth function of its arguments. The aim of this chapter is to prove the following theorem.

Theorem 5.1.1. *Let $s > \frac{d}{2}$ and let $(u_0, u_1) \in H^{s+1} \times H^s$. There exists a time $T > 0$, dependant only on $\|u_0\|_{H^{s+1}}$ and $\|u_1\|_{H^s}$ such that there exists a unique solution to the equation* (5.1.1)

$$u \in C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s).$$

Moreover, the solution depends continuously on the initial data. If $u_0^{(i)}, u_1^{(i)}$ is a sequence of functions which tend to u_0, u_1 in $H^{s+1} \times H^s$ as $i \to \infty$, then the corresponding solution $u^{(i)}$ tends to u in $C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$.

Remark 5.1.2. *When we have such a theorem, we say that the equation is well posed in the space $H^{s+1} \times H^s$.*

In the following, we note $\mathcal{X}_T^s = C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ and

$$\|u\|_{\mathcal{X}_T^s} = \sup_{t \in [0,T]} \left(\|u(t)\|_{H^{s+1}} + \|u(t)\|_{H^s} \right).$$

Let us now prove the theorem.

Proof. We will construct iteratively a sequence of approximations of the solution, and show the convergence of this sequence on a time interval $[0, T]$ where T will be chosen in the process, small enough compared to $\|u_0\|_{H^{s+1}}$ and $\|u_1\|_{H^s}$.

We start with $u^{(0)}$ the solution to the homogeneous equation $\Box u^{(0)} = 0$, with initial data $(u^{(0)}, \partial_t u^{(0)})|_{t=0} = (u_0, u_1)$. Once $u^{(n)}$ is constructed, we take $u^{(n+1)}$ to be the solution of the inhomogeneous linear equation

$$\Box u^{(n+1)} = (\partial_t u^{(n)})^2,$$

with initial data $(u^{(n+1)}, \partial_t u^{(n+1)})|_{t=0} = (u_0, u_1)$.

The strategy is to show that for T small enough :

- the sequence $u^{(n)}$ is uniformly bounded in $\mathcal{X}^s_T$,
- the sequence $u^{(n)}$ is a Cauchy sequence in $\mathcal{X}^s_T$.

We start with the first point. It is sufficient to show the existence of some $A > 0$ satisfying

$$\|u^{(n)}\|_{\mathcal{X}^s_T} \leq A \quad \Rightarrow \|u^{(n+1)}\|_{\mathcal{X}^s_T} \leq A.$$

Using the energy estimate given by Theorem 4.2.8 we can write

$$\|u^{n+1}\|_{\mathcal{X}^s_T} \leq C \left(\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \int_0^T \|(\partial_t u^{(n)})^2\|_{H^s} \right).$$

In the proof, we will denote by C any numerical constant, which may not be the same in every line, the important point being that it does not depend on the functions we consider. We recall that, for $s > \frac{d}{2}$, the Sobolev space H^s is an algebra. Consequently

$$\|(\partial_t u^{(n)})^2\|_{H^s} \leq C \|(\partial_t u^{(n)})\|^2_{H^s} \leq CA^2,$$

and we can write

$$\|u^{n+1}\|_{\mathcal{X}^s_T} \leq C \left(\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + TA^2 \right).$$

By taking

$$A \geq 2C \left(\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} \right)$$

and then choosing T small enough such that

$$TCA \leq \frac{1}{2}$$

we obtain

$$\|u^{n+1}\|_{\mathcal{X}^s_T} \leq A,$$

which prove the first point. Let us not that the condition we have on T at this stage is

$$T \leq \frac{C}{\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s}},$$

where again C is some numerical constant.

Let us now prove the second point. We will show iteratively that

$$\|u^{(n+1)} - u^{(n)}\|_{\mathcal{X}^s_T} \leq \frac{2A}{2^n}.$$

We note that A, which has been defined on the first step, is chosen such that this property is true for $n=0$. Assuming that it is true for $n-1$, we consider the equation satisfied by $u^{(n+1)} - u^{(n)}$:

$$\Box(u^{(n+1)} - u^{(n)}) = (\partial_t u^n)^2 - (\partial_t u^{(n-1)})^2,$$

with zero initial data. Consequently the energy estimate yields

$$\begin{aligned}\|u^{(n+1)} - u^{(n)}\|_{\mathcal{X}_T^s} \leq & C \int_0^T \| \left(\partial_t u^n + \partial_t u^{(n-1)}\right) \left(\partial_t u^n - \partial_t u^{(n-1)}\right) \|_{H^s}. \\ \leq & 2CTA \frac{2A}{2^n}\end{aligned}$$

where we have used again the fact that H^s is an algebra. By taking T such that $2CTA \leq 1$, we can conclude the proof by iteration.

This show that $u^{(n)}$ is a Cauchy sequence in $\mathcal{X}_T^s$, which is a Banach space, so $u^{(n)}$ has a limit u, and this limit satisfies (5.1.1).

$\Box$

5.2 Exercises

Exercise 1.

The aim of this exercise is to show a local existence result for a quasilinear equation, that is to say an equation in which the coefficients of the second order terms depend on the solution itself.

Let $s \in \mathbb{N}$ with $s > 3$. The aim of this exercice is prove that when the parameter $\varepsilon > 0$ is small enough, their exists $T > 0$ such that for all $(u_0, u_1) \in H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with

$$\|(u_0, u_1)\|_{H^{s+1} \times H^s} \leq 1$$

their exists a unique $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ solution of

$$\left\{\begin{array}{l}\Box u = -\varepsilon \left(\partial_{x_1}(u \partial_{x_1} u) + \partial_{x_2}(u \partial_{x_2} u)\right) \\ (u, \partial_t u)|_{t=0} = (u_0, u_1)\end{array}\right. \tag{5.2.1}$$

where $\Box u = -\partial_t^2 u + \Delta u$.

1. Let $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$. We consider the metric g defined by

$$g = -(1+\varepsilon u) dt^2 + (dx_1)^2 + (dx_2)^2 + (1+\varepsilon u)(dx_3)^2.$$

Calculate $\Box_g \phi$ for any function ϕ, and show that $\Box_g u = 0$.

2. Let $m > 0$. We assume that $u, \partial_t u$ are bounded in C^{m+3} . Recall why, for $\varepsilon > 0$ small enough their exists a unique solution $\phi \in C([0,T], H^{m+1}) \cap C^1([0,T], H^m)$ to $\Box_g \phi = 0$ with $(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1) \in H^{m+1} \times H^m$.

3. We consider still the equation $\Box_g \phi = 0$. We note $\nabla^\alpha \phi = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \phi$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Show that we can write

$$\Box \nabla^\alpha \phi + \varepsilon \left(\partial_{x_1}(u \partial_{x_1} \nabla^\alpha \phi) + \partial_{x_2}(u \partial_{x_2} \nabla^\alpha \phi)\right) = F^\alpha$$

with $|F^\alpha| \leq C \sum_{|\beta|+|\gamma| \leq |\alpha|} \sum_{\mu=0}^3 \sum_{\nu=0}^3 |\nabla^\beta \partial_\mu u \nabla^\gamma \partial_\nu \phi|$.

4. Show that

$$\|F^\alpha\|_{L^2(\mathbb{R}^3)} \leq C(\|\partial u\|_{C^{\lfloor \frac{|\alpha|}{2} \rfloor}} \|\partial \phi\|_{H^{|\alpha|}} + \|\partial \phi\|_{C^{\lfloor \frac{|\alpha|}{2} \rfloor}} \|\partial u\|_{H^{|\alpha|}}).$$

5. Deduce that for $|\alpha| \leq s$ we have

$$\|F^\alpha\|_{L^2(\mathbb{R}^3)} \leq C\|\partial u\|_{H^s} \|\partial \phi\|_{H^s}.$$

We assume (u_0, u_1) smooth and compactly supported. We want to construct a sequence $u^{(n)}$ by induction. We take $u^{(0)} = 0$ and $u^{(n+1)}$ to be the solution of

$$\begin{cases} \Box u^{(n+1)} = -\varepsilon \left(\partial_{x_1}(u^{(n)} \partial_{x_1} u^{(n+1)}) + \partial_{x_2}(u^{(n)} \partial_{x_2} u^{(n+1)}) \right) \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{5.2.2}$$

6. Show that for $\varepsilon > 0$ small enough, their exists $T > 0$ such that the sequence is well defined in $C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ and their exists A such that

$$\|(u^{(n)}, \partial_t u^{(n)})\|_{H^{s+1} \times H^s} \leq A.$$

7. Show that, up to the choice of a smaller T, $u^{(n)}$ is a Cauchy sequence in $C([0,T], H^1) \cap C^1([0,T], L^2)$.

8. Conclude (also in the case of (u_0, u_1) only in $H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$).

Exercise 2.

The aim of this exercise is to show that for some non linear equations, when the initial data are sufficiently small, the solution exists for all time.

We consider the equation $\Box u = (\partial_t u)^2$, on $\mathbb{R}^{4+1}$ with initial data $(u, \partial_t u)|_{t=0} = (\varepsilon u_0, \varepsilon u_1)$ with (u_0, u_1) smooth compactly supported functions. The aim of the exercice is to show that their exists ε_0 such that if $0 \leq \varepsilon \leq \varepsilon_0$ the solution exists for all time.

We recall that the vector fields in $\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha,\beta}, S\}$ satidfy $[\Box, Z] = c(Z)\Box$, with $c(Z) = 0$ except $c(S) = -2$. We recall also the Klainerman-Sobolev inequality

$$(1+t+r)^{\frac{n-1}{2}} (1+|r-t|)^{\frac{1}{2}} |f(t,x)| \leq C_n \sum_{|I| \leq \frac{n}{2}+1} \|Z^I f\|_{L^2(\mathbb{R}^n)}.$$

Let $N \geq 6$.

1. Show that their exists $T > 0$, such that the solution u exists on $[0,T] \times \mathbb{R}^4$ and satisfy

$$\sum_{|I| \leq N} \|\partial Z^I u\|_{L^2} \leq 2A_0 \epsilon.$$

where A_0 is such that the above inequality is satisfied at $t = 0$.

Let T be the biggest time satisfying the above condition. Let us assume that $T < \infty$ and find a contradiction for $\varepsilon \leq \varepsilon_0$ (to be determined).

2. Show that for $|I| \leq N-3$ we have

$$(1+t)^{\frac{3}{2}}|\partial Z^I u| \leq CA_0\varepsilon.$$

3. Show that

$$|Z^I(\partial_t u)^2| \leq C \sum_{|J|+|K|\leq|I|} |\partial Z^J u \partial Z^K u|.$$

4. Deduce that for $|I| \leq N$

$$|Z^I(\partial_t u)^2| \leq \frac{C\varepsilon}{(1+t)^{\frac{3}{2}}} \sum_{|J|\leq|I|} |\partial Z^J u|,$$

and

$$\int_0^t \|Z^I(\partial_t u)^2\|_{L^2} \leq C(A_0\varepsilon)^2.$$

5. Conclude.
6. What would be the result in $\mathbb{R}^{3+1}$?

Chapter 6

Choquet Bruhat's theorem

The aim of this final chapter is to show that the initial value problem for Einstein equation is well-posed : given initial data which are sufficiently regular, one can find a unique local solution to Einstein vacuum equation which induce this initial data set. A good intuition of the problem can be obtained with the study of Maxwell equations.

6.1 Preliminaries : Maxwell's equations in Lorentz gauge

We recall Maxwell equations in vacuum

$$\nabla \wedge E = -\frac{\partial B}{\partial t}, \tag{6.1.1}$$

$$\nabla \wedge B = \frac{\partial E}{\partial t}, \tag{6.1.2}$$

$$\nabla . E = 0, \tag{6.1.3}$$

$$\nabla . B = 0. \tag{6.1.4}$$

The initial data, are (E_0, B_0) at time $t = 0$. These data are not arbitrary since the equations (6.1.3) and (6.1.2) should also be satisfied by E_0 and B_0.

To solve Maxwell equation, we introduce the electromagnetic potential. From the equation (6.1.4) we can write that B is the rotational of some vector A :

$$B = \nabla \wedge A.$$

Then from the equation (6.1.1) one can write $\nabla \wedge (E + \frac{\partial A}{\partial t}) = 0$, and this yields

$$E = -\frac{\partial A}{\partial t} + \nabla V,$$

for some function V. We now give the equation (6.1.2) and (6.1.3) in term of A and V

$$-\frac{\partial}{\partial t}\nabla . A + \Delta V = 0$$

$$\nabla(\nabla . A) - \Delta A = -\frac{\partial^2 A}{\partial t^2} + \frac{\partial}{\partial t}\nabla V.$$

There is a gauge freedom in the choice of A, V. Indeed a transformation of the form

$$A \to A + \nabla\chi, \qquad V \to V + \partial_t \chi$$

does not change the value of E and B. We can use this freedom to impose a condition on A and V to write the equation in a more tractable form. A possibility is to work in Lorentz gauge, that is to say under the condition

$$\nabla.A = \partial_t V. \tag{6.1.5}$$

Under this condition, the equation for A and V are simply

$$\Box A = 0, \qquad \Box V = 0. \tag{6.1.6}$$

The strategy to solve Maxwell equations in Lorentz gauge is the following.

- The physical initial data are (E_0, B_0) at time $t = 0$, satisfying the constraints (6.1.3) and (6.1.4). The initial data for the equations the potential, $(V, \partial_t V)|_{t=0}$ and $(A, \partial_t A)|_{t=0}$ are chosen such that
  - $B_0 = \nabla \wedge A$,
  - V is free,
  - $E_0 = -\partial_t A + \nabla V$
  - The Lorentz gauge condition is satisfied at $t = 0$: $\partial_t V = \nabla.A$.
- We solve (6.1.6) with these initial data.
- We show that we have indeed constructed a solution to Maxwell equation. For this, what we need to show is that the Lorentz condition (6.1.5) remains true for all time. We note that we have

  $$\Box(\partial_t V - \nabla.A) = 0.$$

  Therefore, by unicity of the solutions to the wave equation, it is sufficient to show that initially, we have

  $$\partial_t V - \nabla.A = 0 \tag{6.1.7}$$
  $$\partial_t(\partial_t V - \nabla.A) = 0. \tag{6.1.8}$$

  The first condition is ensured by the choice of $\partial_t V$ at time $t = 0$. For the second, we use the equation $\Box V = 0$, and the initial condition on $\partial_t A$ to write

  $$\partial_t(\partial_t V - \nabla.A) = \nabla.(\nabla V - \partial_t A) = \nabla.E_0.$$

  Since E_0 is assumed to satisfy the constraint $\nabla.E_0 = 0$, the initial condition (6.1.8) is satisfied. The Lorentz condition is true for all time, and we have solved Maxwell equations.

6.2 Local well posedness for Einstein equations

We recall that the initial data for Einstein equation are a triplet $(\Sigma, \bar{g}, K)$ with Σ a 3-dimensional manifold, $\bar{g}$ a Riemannian metric on Σ and K a symmetric 2-tensor. Solving Einstein equations with these data consist in finding $(\mathcal{M}, g)$ such that

$$\Sigma \subset \mathcal{M}, \quad g|_\Sigma = \bar{g},$$

and K is the second fundamental form of the embedding of Σ in $\mathcal{M}$. The initial data are not arbitrary. Indeed, we have the following corollary of Proposition 3.2.3

Corollary 6.2.1. *The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ satisfies*

$$G_{0j} = N(\partial_j(tr_{\bar{g}}K) - \overline{\nabla}^h K_{hj})$$
$$G_{00} = \frac{1}{2}(\bar{R} + (tr_{\bar{g}}K)^2 - |K|^2),$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$.

Consequently, the equations $G_{0i} = 0$ and G_{00} only involve $\bar{g}$ and K and should be satisfied by the initial data. They can be seen as the equivalent of (6.1.3) and (6.1.4) for Maxwell equations.

The Theorem we will be interested in is the following, due to Choquet-Bruhat [5] for the local existence part, and to Choquet-Bruhat and Geroch [2] for the uniqueness part.

Theorem 6.2.2. *Let $(\Sigma, \bar{g}, K)$ be smooth initial data, satisfying the constraint equations. Their exists a unique, maximal, globally hyperbolic solution to the Einstein equation, (M, g) corresponding to these initial data.*

Let us explain a little this theorem.

- By globally hyperbolic, we mean that every curve which is timelike and inextendible should intersect Σ, which is an hypersurface of M. We say also that Σ is a Cauchy hypersurface for M.
- By unique and maximal, we mean that every other solution with these initial data can be isometrically embedded in M.

We will give the main ideas of the proof of this theorem, which is based on the use of wave coordinates.

6.3 Einstein equations in wave coordinates

We have the following lemma.

Lemma 6.3.1. *In any coordinate system x^α, the Ricci tensor can be written*

$$R_{\nu\nu} = -\frac{1}{2}\Box_g g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + P_{\mu\nu}(g)(\partial g, \partial g),$$

where

$$H^\rho = \Box_g x^\rho = \frac{1}{\sqrt{|det(g)|}}\partial_\alpha(g^{\alpha\rho}\sqrt{|det(g)|}),$$

and $P_{\mu\nu}(g)(\partial g, \partial g)$ is a quadratic form in the first derivatives of g.

Proof. We recall the expression of the Ricci tensor in a coordinate system

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha},$$

and the expression for the Christophel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho}(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}).$$

Therefore, we can compute (as in the exercise on the linearized Einstein equations)

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta}(\partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\nu\partial_\beta g_{\alpha\mu}) - \frac{1}{2} g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} + \widetilde{P}_{\mu\nu}(g)(\partial g, \partial g),$$

where $\widetilde{P}_{\mu\nu}(g)(\partial g, \partial g)$ stand for the quadratic terms. We note that $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$ is equal to $\Box_g g_{\mu\nu}$ plus a quadratic term in ∂g. Consequently, if we write

$$H^\alpha = \partial_\beta g^{\beta\alpha} - \frac{1}{2} g^{\beta\rho}\partial^\alpha g_{\beta\rho},$$

we can write

$$R_{\nu\nu} = -\frac{1}{2}\Box_g g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + P_{\mu\nu}(g)(\partial g, \partial g).$$

We also note that

$$H^\rho = -g^{\alpha\beta}\Gamma^\rho_{\alpha\beta} = \nabla^\alpha \partial_\alpha x^\rho = \Box_g x^\rho.$$

This concludes the proof of the lemma. □

The wave coordinate condition consists in taking $H^\rho = 0$, to remove all the second order term in the equations, except the one which can be written as a wave equation. If $H^\rho = 0$, then Einstein equations can be written

$$\Box_g g_{\mu\nu} = P_{\mu\nu}(\partial g, \partial g). \tag{6.3.1}$$

This is analogous to the Lorentz gauge for Maxwell equations. The strategy to solve Einstein equation in wave coordinates is the following.

- The physical initial data are $(\bar{g}, K)$. To solve (6.3.1) we need the initial data for $g_{\mu\nu}$ and $\partial_t g_{\mu\nu}$ at time $t = 0$. We choose them with the following process.
  - We take $g_{ij} = \bar{g}_{ij}$,
  - g_{00} and g_{0i} are free : we can choose $g_{00} = -1$ and $g_{0i} = 0$ without loss of generality,
  - We take $\partial_t g_{ij} = K_{ij}$.
  - We choose $\partial_t g_{00}$ and $\partial_t g_{0i}$ in order for the wave coordinate condition to be satisfied at time $t = 0$.
- We solve (6.3.1) with this initial data. Since it is a nonlinear equation, the time of existence T is a priori only finite. As the semilinear model we have studied, the equation (6.3.1) is well posed in $H^{s+1} \times H^s$ with $s > \frac{3}{2}$, under the additional condition that $|g_{ij} - \delta_{ij}| \leq \frac{1}{8}$.
- We now have to check that the metric g, whose coefficients are given by the solution of (6.3.1) is indeed a solution to Einstein equations.

Let us describe this last point. What we need to show is that the wave coordinate condition is true as long as the solution exists. We consider the metric g we have constructed : we can calculate its Ricci tensor

$$\begin{aligned} R_{\nu\nu} &= -\frac{1}{2}\Box_g g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + P_{\mu\nu}(g)(\partial g, \partial g), \\ &= \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right), \end{aligned}$$

where we have used the wave equations satisfied by g. Let us now recall the contracted Bianchi identity, which are always satisfied by the Einstein tensor of a metric

$$\nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0.$$

Replacing the Ricci tensor by its expression in term of H, we obtain for H a system of linear, homogeneous wave equations with variable coefficients. By unicity of the solution, it is therefore sufficient to check that the initial data vanish at time $t = 0$. The condition $H^\rho|_{t=0} = 0$ is ensured by the choice of $\partial_t g_{00}$ and $\partial_t g_{0i}$. To obtain the remaining condition, $\partial_t H^\rho|_{t=0} = 0$, as for Maxwell equations, we look at the constraint equations. We have that, at time $t = 0$

$$R_{00} - \frac{1}{2}Rg_{00} = 0, \qquad R_{0i} = 0.$$

Again, the Ricci tensor can be expressed in term of ∂H, and here more precisely in term of $\partial_t H$, since H vanishes at time zero. The condition we obtain on $\partial_t H$ can be inverted, yielding the desired initial condition $\partial_t H^\rho_{t=0} = 0$.

So far, we have prove the local existence of solutions when there is a global coordinate chart on Σ on which the initial data are close enough to the Minkowski metric. In the general case, we can consider, for any point $p \in \Sigma$, a coordinate chart around p in which at p, $g_{ij} = \delta_{ij}$. Working in a smaller neighbourhood of p if necessary, it is possible to assume that the space-time metric is close to the Minkowski metric. Then, the strategy is to cover Σ by such neighbourhood, and show that in the intersection of two neighbourhoods, the constructed solutions are isometric to each other.

6.4 Exercises

Exercise 1.

The aim of this exercise is to construct approximate gravitational wave solutions to Einstein equation. We study solutions of vacuum Einstein equations of the form $g_{\alpha\beta} = m_{\alpha\beta} + \gamma_{\alpha\beta}$ where $\gamma_{\alpha\beta}$ is small. We recall that the linearized Einstein equations around Minkowski metric are

$$\delta G_{\mu\beta} = -\frac{1}{2}\partial^\alpha\partial_\alpha\bar{\gamma}_{\mu\beta} + \frac{1}{2}\partial^\alpha(\partial_\beta\bar{\gamma}_{\alpha\mu} + \partial_\mu\bar{\gamma}_{\beta\alpha}) - \frac{1}{2}m_{\mu\beta}\partial^\alpha\partial^\rho\bar{\gamma}_{\alpha\rho} = 0,$$

where $\bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{1}{2}\gamma m_{\alpha\beta}$. These equation are invariant by the transformations $\bar{\gamma}_{\alpha\beta} \to \bar{\gamma}_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha - \partial^\mu\xi_\mu m_{\alpha\beta}$ for all vector field ξ.

1. Write the linearized constraint equations : $\delta G_{00} = 0$, $\delta G_{0i} = 0$.

2. Remember why one can choose ξ such that $\partial^\beta\bar{\gamma}_{\alpha\beta} = 0$. How do you write the linearized Einstein equations in this gauge ?

3. $(\xi, \partial_t \xi)|_{t=0}$ can be choosen freely. What would be the equation for them to have initially $\gamma = \partial_t \gamma = 0$ and $\gamma_{0i} = \partial_t \gamma_{0i} = 0$?

4. Find some transformation to show that these equations on γ can be solved (having in mind that we know how to solve an equation of the form $\Delta u = f$).

5. Write the linearized constraint equations in term of γ_{ij} and $\partial_t \gamma_{ij}$.

6. Using the same method as for Maxwell equation, or Eintein equation in wave coordinate, show that the linearized Einstein equations can be solved in the wave gauge, with $\gamma = 0$ and $\gamma_{0i} = 0$ everywhere. Such a gauge is called the radiation gauge.

7. Using the equation $\delta R_{00} = 0$, show that we must have $\gamma_{00} = 0$ everywhere.

8. We now seek plane wave solutions $\gamma_{ab} = H_{ab} e^{i k_\mu x^\mu}$, where H_{ab} is a constant tensor field and k_μ a constant 1 form. What conditions should they satisfy to have a solution of the linearized Einstein equations in radiation gauge ?

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