Stability in exponential time of Minkowski space-time with a space-like translation symmetry

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Abstract

In this note, we discuss the nonlinear stability in exponential time of Minkowki space-time with a translation space-like Killing field, proved in [13]. In the presence of such a symmetry, the $3+1$ vacuum Einstein equations reduce to the $2+1$ Einstein equations with a scalar field. We work in generalized wave coordinates. In this gauge Einstein equations can be written as a system of quasilinear quadratic wave equations. The main difficulty in [13] is due to the decay in $\frac{1}{\sqrt{t}}$ of free solutions to the wave equation in 2 dimensions, which is weaker than in 3 dimensions. As in [21], we have to rely on the particular structure of Einstein equations in wave coordinates. We also have to carefully chose an approximate solution with a non trivial behaviour at space-like infinity to enforce convergence to Minkowski space-time at time-like infinity.

1 Introduction

1.1 Einstein equations

The equations of general relativity, introduced by Einstein in 1915, link the geometry of space-time to the distributions of matter and fields present in the universe. The space-time is described by a 4 dimensional manifold $\mathcal{M}$, equipped with a Lorentzian metric g, that is to say a metric of signature $(-1,1,1,1)$. Einstein equations can be written

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = T_{\mu\nu}. \quad (1)$$

The right-hand side $T_{\mu\nu}$ is the energy-momentum tensor, which describes the masses, electromagnetic fields and other physical fields arising in the universe. The left-hand side describes the curvature of our space-time $(\mathcal{M}, g)$. More precisely, $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature, which is defined by the trace of the Ricci tensor. The Ricci tensor is a nonlinear second order operator acting on the metric g.

In the vacuum case, $T_{\mu\nu} = 0$, Einstein equations can be written

$$R_{\mu\nu} = 0. \quad (2)$$

A trivial solution in this case is given by Minkowski space-time $\mathbb{R}^{3+1}$ equipped with the Minkowski metric

$$m = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

This special solution is a flat metric. Its Riemann curvature tensor is zero, and consequently its Ricci curvature tensor is zero. However, in dimension greater or equal to 4, not all the solutions of (2) are flat. An efficient way to study the set of solutions of (2) is to formulate Einstein equations as a Cauchy problem.

1.2 The Cauchy problem

The Cauchy data for Einstein equations are given by a triplet $(\Sigma, \bar{g}, K)$, where Σ is a 3 dimensional manifold, $\bar{g}$ is a Riemannian metric on Σ and K is a symmetric 2-tensor, which is heuristically the data of $\partial_t \bar{g}$. Solving Einstein equations with initial data $(\Sigma, \bar{g}, K)$ consists in finding a space-time $(\mathcal{M}, g)$ satisfying (2) such that $\Sigma \subset \mathcal{M}$, $\bar{g}$ is the restriction of g to Σ, and K is the second fundamental form of the embedding of Σ into $\mathcal{M}$.

The initial data cannot be chosen arbitrarily. There are compatibility conditions, known as the constraint equations, that the initial data $(\Sigma, \bar{g}, K)$ must satisfy. More precisely, the equations

$$R_{0i} = 0, \quad R_{00} - \frac{1}{2} R g_{00} = 0,$$

can be expressed in terms only of the initial data. They can be written

$$\begin{aligned} \bar{R} - K_{ij}K^{ij} + (K^h{}_h)^2 &= 0, \\ \partial_j(K^h{}_h) - D_h K^h{}_j &= 0, \end{aligned}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$ and D is the Levi-Civita connection associated to $\bar{g}$.

The constraint equations are a necessary and sufficient condition on the initial data for the existence of local solutions, as proven in the pioneer result of Choquet-Bruhat and Geroch.

Theorem 1.1 ([7])**.** *For initial data $(\Sigma, \bar{g}, K)$ sufficiently smooth and solutions of the constraint equations, there exists a unique maximal globally hyperbolic developpement, solution of Einstein vacuum equations.*

Finding solutions to the constraint equations is a research area in itself (see the review [4]).

1.3 Stability of Minkowski

Theorem 1.1 is a local result. Due to the nonlinear character of Einstein equations, the local solutions do not in general extend globally. What could be expected is the stability of special solutions. In this setting, a fundamental question is the stability of Minkowski solution.

The initial data for Minkowski are $(\mathbb{R}^3, e, 0)$ where e is the Euclidean metric. In [9], Christodoulou and Klainerman proved the following : for initial data $(\mathbb{R}^3, \bar{g}, K)$ sufficiently smooth, such that $\bar{g}$ is closed to e, K small, and asymptotically flat ($\bar{g}$ tend to e and K tend to 0 with a specified decay rate), the Cauchy development is geodesically complete, and the solution converges at infinity to Minkowski solution. An other proof of the stability has been given later by Lindblad and Rodnianski in harmonic gauge (see [21]).

1.4 Einstein equations with a translation symmetry

Einstein equations being a quite difficult nonlinear system, one way to study them and have a better understanding of their structure is to introduce some symmetries. The translation symmetry, studied by Choquet-Bruhat and Moncrief in [8] allows to reduce the $3+1$ dimensional problem to a $2+1$ dimensional one. More precisely, we look for solutions of the $3+1$ vacuum Einstein equation, on manifolds of the form $\Sigma \times \mathbb{R}_{x_3} \times \mathbb{R}_t$, where Σ is a 2 dimensional manifold, equipped with a metric of the form

$$\mathbf{g} = e^{-2\phi} g + e^{2\phi}(dx^3)^2,$$

where ϕ a scalar function, and g a Lorentzian metric on $\Sigma \times \mathbb{R}$, all quantities being independent of x^3. For these metrics, Einstein vacuum equations are equivalent to the $2+1$ dimensional system

$$\begin{cases} \Box_g \phi = 0 \\ R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi, \end{cases} \tag{3}$$

where $R_{\mu\nu}$ is the Ricci tensor associated to g. Choquet-Bruhat and Moncrief studied the case where Σ is compact of genus $G \geq 2$. In [8], they proved the stability of a particular expending solution. Here we will work in the case $\Sigma = \mathbb{R}^2$. Then a particular solution is given by Minkowski solution itself. It corresponds to $\phi = 0$ and g equals to the Minkowski metric in dimension $2+1$. A natural question one can ask is the stability of this solution.

In [13] we prove the existence of solutions in exponential time : for initial data for ϕ in some weighted Sobolev spaces, of size ε small, there exist solutions to (3) for times $t \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$. We recall the definition of weighted Sobolev spaces

$$\|u\|_{H^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2}.$$

Theorem 1.2. *Let $0 < \varepsilon < 1$. Let $N \geq 40$, $\frac{1}{2} \leq \delta \leq 1$ and $0 < \rho < \frac{1}{2}$. Let $(\phi_0, \phi_1) \in H^{N+1}_\delta \times H^N_{\delta+1}$ such that*

$$\|\phi_0\|_{H^{N+1}_\delta} + \|\phi_1\|_{H^N_{\delta+1}} = \varepsilon$$

There exists a constant C such that if $T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$ and ε is small enough, there exist a coordinate system (t, x_1, x_2) and a solution (ϕ, g) of (3) on $[0,T] \times \mathbb{R}^2$ such that

$$(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1),$$

and we have the estimates

$$|g_{\alpha\beta} - m_{\alpha\beta}| \lesssim \varepsilon,$$

$$|g_{\alpha\beta} - m_{\alpha\beta}| \lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}-\rho}}, \quad \text{for } r \leq \frac{t}{2},$$

where $m_{\alpha\beta}$ is Minkowski metric on $\mathbb{R}^{2+1}$.

For a more precise statement of Theorem 1.2, we refer to [13].

Comments on this theorem

- The initial data for g must satisfy the constraint equations. The only freedom in this solving is the choice of the initial hypersurface. The construction of solutions to the constraint equations for this problem is done in [12].
- The perturbations we consider are not asymptotically flat in $3+1$ dimension, since asymptotic flatness is not compatible with a translation spacelike symmetry.
- The method used to prove this theorem is by using a wave gauge. In this sense it is similar to Lindblad and Rodnianski proof of the stability of Minkowski.

Outline of this note

- In Section 2, we present the wave coordinate condition, which is a gauge choice which allows to write Einstein equations as a system of quasilinear wave equations. Then we present the methods for proving long-time existence for small data for such systems of nonlinear wave equations. We then exhibit more precisely the structure of Einstein equations in wave coordinates.

- In Section 3 we explain how to construct an approximate solution for our system. To give an intuition, we present first a family of exact radial solutions : Einstein-Rosen waves. We then explain how to adapt the analysis in the non-radial case. We present the new choice of coordinates which we need to make to be compatible with the constructed approximate solution.
- In Section 4 we give a brief outline of the proof of Theorem 1.2, focusing on the main technical tools which are needed : weighted energy estimate, $L^\infty - L^\infty$ estimate. We also explain how the constructed approximate solution is helpful in proving Theorem 1.2, and how it leads to the restriction to exponential time.

2 Einstein equations in wave coordinates

2.1 The choice of coordinates

In a coordinate system x^α, the Ricci tensor is given by

$$R_{\mu\nu} = -\frac{1}{2}g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\mu\nu} + \frac{1}{2}H^\rho\partial_\rho g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + \frac{1}{2}P_{\mu\nu}(g)(\partial g, \partial g), \tag{4}$$

where $P_{\mu\nu}(g)(\partial g, \partial g)$ is a quadratic form in ∂g and

$$H^\alpha = \Box_g x^\alpha = -\partial_\lambda g^{\lambda\alpha} - \frac{1}{2}g^{\lambda\mu}\partial^\alpha g_{\lambda\mu}. \tag{5}$$

The wave coordinate condition (respectively the generalized wave coordinate condition) consists in imposing $H^\alpha = 0$ (respecively $H^\alpha = F^\alpha$ a fixed function, which may depend on g but not on its derivatives).

Proposition 2.1. *If the coupled system of equations*

$$\begin{cases} -\frac{1}{2}g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\mu\nu} + \frac{1}{2}F^\rho\partial_\rho g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu F^\rho + g_{\nu\rho}\partial_\mu F^\rho\right) + \frac{1}{2}P_{\mu\nu}(g)(\partial g, \partial g) = 2\partial_\mu\phi\partial_\nu\phi \\ g^{\alpha\rho}\partial_\alpha\partial_\rho\phi - F^\rho\partial_\rho\phi = 0 \end{cases} \tag{6}$$

with F a function which may depend on ϕ, g, is satisfied on a time interval $[0,T]$ with $T > 0$, if the initial induced riemannian metric and second fundamental form $(\bar{g}, K)$ satisfy the constraint equations, and if the initial compatibility condition

$$F^\alpha|_{t=0} = H^\alpha|_{t=0}, \tag{7}$$

is satisfied, then the equations (3) *are satisfied on* $[0,T]$*, together with the wave coordinate condition*

$$F^\alpha = H^\alpha.$$

For a proof of this result, we refer to [22], or the appendix of [13].

2.2 Long-time existence problem for system of nonlinear wave equations

The aim of this section is to give some methods and results on the study of equations of the type $\Box u = (\partial u)^p$.

We consider first the equation in $\mathbb{R}^{n+1}$

$$\begin{cases} \Box u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{8}$$

By multiplying this equation by $\partial_t u$ and integrating over $\mathbb{R}^2$ we obtain the conservation of energy

$$\int_{\mathbb{R}^2} (\partial_t u)^2 + |\nabla u|^2 = \int u_1^2 + |\nabla u_0|^2. \tag{9}$$

To obtain point-wise decay from the energy estimate, we introduce the following family of vector fields which are isometries and conformal isometries of Minkowski space-time :

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha \partial_\beta + x_\beta \partial_\alpha, S = t\partial_t + r\partial_r\}.$$

These vector fields satisfy the following commutation property :

$$[\Box, Z] = C(Z)\Box,$$

where

$$C(Z) = 0,\ Z \neq S, \quad C(S) = 2.$$

Therefore if $\Box u = 0$ then $\Box Z u = 0$ and the energy estimate (9) yields

$$\int \left(\left(\frac{d}{dt} Z^I u \right)^2 + |\nabla Z^I u|^2 \right)(t,x)dx = \int \left(\left(\frac{d}{dt} Z^I u \right)^2 + |\nabla Z^I u|^2 \right)(0,x)dx \tag{10}$$

where $Z^I u$ denotes any combination of I vector fields of $\mathcal{Z}$.

The following estimate, called Klainerman-Sobolev inequality gives a more precise information than Sobolev embedding $H^s \subset L^\infty$ for $s > \frac{n}{2}$, providing we control the L^2 norms of $Z^I u$ (which results from (10)). It can be written

$$(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{1}{2}} |v(t,x)| \leq C \sum_{|I| \leq \frac{n}{2}+1} \|Z^I v\|_{L^2}. \tag{11}$$

Thanks to this inequality we recover the decay rate $u \sim t^{-\frac{n-1}{2}}$ for a solution of $\Box u = 0$. More over, a simple calculation gives (we note $r = |x|$)

$$\partial_t + \partial_r = \frac{S + \sum_{i=1}^n \frac{x_i}{r} \Omega_{0i}}{t + r},$$
$$\partial_t - \partial_r = \frac{S - \sum_{i=1}^n \frac{x_i}{r} \Omega_{0i}}{t - r}.$$

Consequently, the derivative tangential to the light cone, that we will denote $\bar{\partial}$, have a better decay rate given by $\bar{\partial} u \sim t^{-\frac{n+1}{2}}$.

We now consider a nonlinear problem

$$\begin{cases} \Box u = (\partial u)^p, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{12}$$

with initial data (u_0, u_1) of size ε small. The energy estimate yields

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^n} (\partial u)^2(t,0)dx + \int_0^t \int (\partial u)^{p+1}(x,s)dxds.$$

If we suppose a priori estimates compatible with the linear case

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq C\varepsilon, \quad |\partial u| \leq \frac{C\varepsilon}{(1+t)^{\frac{n-1}{2}}},$$

we obtain

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^n} (\partial u)^2(t,0)dx + C^{p+1}\varepsilon^{p+1} \int_0^t \frac{1}{(1+s)^{(p-1)\frac{n-1}{2}}} ds,$$

Therefore, if $(p-1)\frac{n-1}{2} > 1$, the space-time integral in the right-hand side converges, which is the first step to prove global existence (see for example [16] for the case $n > 3$ and $p = 2$). In the opposite case, there are counter-examples to global existence (see [14]).

When the nonlinearity has structure, it is possible to obtain a better result. Let's assume for example $n = 3$ and $p = 2$ Then we have $(p-1)\frac{n-1}{2} = 1$, and there is not always global existence. However if we can write the nonlinearity under the form $\partial u \bar{\partial} u$, the energy estimate, together wih the a priori estimates on tangential derivatives

$$|\bar{\partial} u| \leq \frac{C\varepsilon}{(1+t)^2},$$

yields

$$\int_{\mathbb{R}^3} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^3} (\partial u)^2(t,0)dx + C^2\varepsilon^2 \int_0^t \int \frac{1}{(1+s)^2} dxds,$$

which suggest global existence. It is the case for systems of the form

$$\Box u^i = P^i(\partial u^j, \partial u^k), \tag{13}$$

where the P^i satisfy the null condition, introduced by Klainerman in [15]. This condition consists in saying that the P_i are linear combinations of the following forms

$$Q_0(\partial u, \partial v) = \partial_t u \partial_t v - \nabla u.\nabla v, \quad Q_{\alpha\beta}(\partial u, \partial v) = \partial_\alpha u \partial_\beta v - \partial_\alpha v \partial_\beta u.$$

In $2+1$ dimensions, to show global existence, one has to be careful with both quadratic and cubic terms. Quasilinear scalar wave equations in $2+1$ dimensions have been studied by Alinhac in [1]. He shows global existence for a quasilinear equation of the form

$$\Box u = g^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u,$$

if the quadratic and cubic terms in the right-hand side satisfy the null condition. Global existence for a semi-linear wave equation with the quadratic and cubic terms satisfying the null condition has been shown by Godin in [10] using an algebraic trick to remove the quadratic terms, which does however not extend to systems. The global existence in the case of systems of semi-linear wave equations with the null structure has been shown by Hoshiga in [11]. It requires the use of $L^\infty - L^\infty$ estimates for the inhomogeneous wave equations, introduced in [17] (see also Proposition 4.3).

The null condition is not a necessary condition to obtain global existence. An example is given by the following system

$$\begin{cases} \Box \phi_1 = 0, \\ \Box \phi_2 = (\partial_t \phi_1)^2. \end{cases} \tag{14}$$

The decoupling allows to solve (14). However, ϕ_2 has the decaying rate $\phi_2 \sim t^{-1}\ln(t)$, which is less than the behaviour in t^{-1} for the corresponding linear wave equation. It is the same for the equation

$$\partial_t^2 u - u\Delta u,$$

studied in [18], [2] and [19]. These two examples possess a structure, introduced in [20] called weak null structure. Lindblad and Rodnianski proved the stability of Minkowski space-time in wave coordinates by showing that Einstein equations in these coordinates have the weak null structure (see [21]). The proof of theorem 1.2 is also based on this weak null structure

2.3 Structure of Einstein equations in wave coordinates

The structure of Einstein equations can be seen when we write them in the null frame $L = \partial_s$, $\underline{L} = \partial_q$, $U = \frac{\partial_\theta}{r}$, where (r, θ) are the polar coordinates and $q = r - t$ and $s = r + t$ are the null coordinates. We decompose the metric under the form

$$g = m + \widetilde{g} + g_{\underline{LL}} dq^2,$$

where m is the Minkowski metric. Then, if we neglect all the nonlinearities involving a good derivative, we obtain the following model system for (3) in wave coordinates

$$\begin{cases} \Box\phi + g_{LL}\partial_q^2\phi = 0, \\ \Box\widetilde{g} + g_{LL}\partial_q^2\widetilde{g} = 0, \\ \Box g_{\underline{LL}} + g_{LL}\partial_q^2 g_{\underline{LL}} = -4(\partial_q\phi)^2. \end{cases}$$

The quadratic terms involving g_{LL} are handled by making use of the wave coordinate condition, as in [21] : the condition $H^\alpha = 0$ where H^α is defined by (5) implies $\partial_q g_{LL} \sim \bar{\partial}\widetilde{g}$. Therefore, the quadratic terms involving g_{LL} behave like terms having the null structure. Consequently, we are left with the model sustem

$$\begin{cases} \Box\phi = 0, \\ \Box g_{\underline{LL}} = -4(\partial_q\phi)^2. \end{cases}$$

Thanks to the decoupling it is of course possible to solve such a system. However, in $2+1$ dimensions, for initial data of size ε the energy estimate yields

$$\|\partial g_{\underline{LL}}\|_{L^2} \lesssim \varepsilon\sqrt{1+t},$$

and the metric coefficient $g_{\underline{LL}}$ has no decay, not even with respect to $q = r - t$. This is not enough to solve the full coupled system. To prove Theorem 1.2 it will be important to look more precisely at the behaviour which can be expected for our solutions.

3 Construction of approximate solutions

To understand what behaviour may be expected for our solutions, we will look at special solutions of vacuum Einstein equations with a translation space-like Killing field : Einstein-Rosen waves. These solutions have been discovered by Beck (see [5], and also [3] and [6] for a mathematical description).

3.1 Einstein-Rosen waves

Einstein-Rosen waves are solutions of vacuum Einstein equations with two space-like orthogonal Killing fields : ∂_3 and ∂_θ. The $3+1$ metric can be written

$$\mathbf{g} = e^{2\phi}(dx^3)^2 + e^{2(a-\phi)}(-dt^2 + dr^2) + e^{-2\phi}r^2 d\theta^2.$$

The reduced equations

$$\begin{cases} R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi, \\ \Box_g\phi = 0, \end{cases}$$

can be written in this setting

$$R_{tt} = \partial_r^2 a - \partial_t^2 a + \frac{1}{r}\partial_r a = 4(\partial_t\phi)^2, \tag{15}$$

$$R_{rr} = -\partial_r^2 a + \partial_t^2 a + \frac{1}{r}\partial_r a = 4(\partial_r\phi)^2,$$

$$R_{tr} = \frac{1}{r}\partial_t a = 4\partial_t\phi\partial_r\phi.$$

The equation for ϕ can be written, since ϕ is radial

$$e^{2a}\Box_g \phi = -\partial_t^2 \phi + \partial_r^2 \phi + \frac{1}{r}\partial_r \phi = 0,$$

where g is the metric

$$g = e^{2a}(-dt^2 + dr^2) + r^2 d\theta^2.$$

The equation for ϕ decouples from the equations for the metric. Therefore we can solve the flat wave equation $\Box \phi = 0$, with initial data $(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1)$ and then solve the Einstein equations, which reduces to

$$\partial_r a = 2r\left((\partial_r \phi)^2 + (\partial_t \phi)^2\right), \tag{16}$$

with the boundary condition $\phi|_{r=0} = 0$ in order to have a smooth solution. Since $\Box\phi = 0$, if (ϕ_0, ϕ_1) have enough decay, we have the following decay estimate for ϕ

$$|\partial \phi(r,t)| \lesssim \frac{1}{\sqrt{1+t+r}(1+|t-r|)^{\frac{3}{2}}}.$$

Therefore since

$$a = 2\int_0^R r\left((\partial_r \phi)^2 + (\partial_t \phi)^2\right) dr$$

we have

$$|a| \lesssim \frac{1}{(1+|r-t|)^2},\ for\ r < t,$$
$$|a - 2E(\phi)| \lesssim \frac{1}{(1+|r-t|)^2},\ for\ r > t,$$

where the energy

$$E(\phi) = \int_0^\infty r\left((\partial_r \phi)^2 + (\partial_t \phi)^2\right) dr$$

does not depend on t. For $r > t$, we have $a \sim E(\gamma)$ and hence is only bounded. In particular, the metric

$$e^{2a} dr^2 + r^2 d\theta^2$$

exhibits a deficit angle at space-like infinity, that is to say the circles of radius r have a perimeter growth of $e^{-2E(\phi)} 2\pi r$ instead of $2\pi r$. However, in the interior, the decay we get is far better than the one we could have found with standard estimates, if we had used (15) instead of (16).

3.2 Asymptotic behaviour

We would like to adapt the analysis of Section 3.1 in the case where we only assume one Killing field (i.e. in the case where ∂_3 is Killing but not ∂_θ). Let assume that Einstein-Rosen waves are still approximate solutions to (3). As in this case ϕ also depends on θ, we will have

$$\lim_{R\to\infty} a(t,R,\theta) = \int_0^\infty r\left((\partial_r \phi)^2 + (\partial_t \phi)^2\right) dr = b(t,\theta).$$

We have to be careful with the dependence on θ. The metric

$$e^{2b(\theta)}(-dt^2 + dr^2) + r^2 d\theta^2$$

is no longer a Ricci flat metric when b depends on θ. Consequently it is not a good guess for the behaviour at infinity of our metric solution g. A good candidate should be Ricci flat in the region $r > t$. Indeed if we considered compactly supported initial data for ϕ, by finite speed of propagation, ϕ should intuitively be supported in the region $r < t$. Consequently, the equation

$$R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi$$

would imply that g should be Ricci flat for $r > t$.

To get an intuition of the behaviour of our solution without the additional rotational symmetry, let's assume for a moment we had found a coordinate system (not the wave gauge) in which all the metric coefficients have at least the decay of a solution to the free wave equation. Then we can compute, on the light cone

$$R_{\underline{L}\underline{L}} = -\partial_q^2 g_{UU} + O\left(\frac{\varepsilon}{(1+r)^{\frac{3}{2}}}\right).$$

Since we also have

$$R_{\underline{L}\underline{L}} = (\partial_q \phi)^2 = O\left(\frac{\varepsilon}{1+r}\right),$$

we see that the only term which could balance this behaviour is $\partial_q^2 g_{UU}$. This leads to introducing the following family of metrics

$$g_b = -dt^2 + dr^2 + (r + \chi(q)b(\theta)q)^2 d\theta^2, \tag{17}$$

where (r,θ) are polar coordinates, $q = r - t$, χ is a cut-off function such that $\chi(q) = 0$ for $q < 1$ and $\chi(q) = 1$ for $q > 2$, and $b(\theta)$ is a function of θ that we would like to satisfy

$$b(\theta) = \int_{\Sigma_{T,\theta}} (\partial_q \phi)^2 r dq,$$

where $\Sigma_{T,\theta}$ is the half line of fixed angle θ in the hypersurface $t = T$. A calculation yields that all the Ricci coefficients of g_b are zero except

$$(R_b)_{\underline{L}\underline{L}} = -\frac{b(\theta)\partial_q^2(q\chi(q))}{r + b(\theta)q\chi(q)} \tag{18}$$

Therefore, the metrics g_b are Ricci flat in the region $r > t+2$, and correspond to Minkowski metric in the region $r < t+1$.

3.3 The generalized wave coordinates

This choice of background metric will force us to work in generalized wave coordinates, instead of usual wave coordinates. Indeed, for the metric g_b defined by (17), the coordinates (t, x_1, x_2) are not wave coordinates, not even asymptotically. We will choose coordinates x^α such that

$$\Box_g x^\alpha = H_b^\alpha \underset{def}{=} \Box_{g_b} x^\alpha.$$

We look for solutions of the form $g = g_b + \widetilde{g}$. Then (3) can be written

$$\begin{cases} \Box_g \phi = 0, \\ \Box_g \widetilde{g}_{\mu\nu} = -2\partial_\mu \phi \partial_\nu \phi + 2(R_b)_{\mu\nu} + P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g}) + \widetilde{P}_{\mu\nu}(\widetilde{g}, g_b), \end{cases} \tag{19}$$

where $P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g})$ is a quadratic form in $\partial\widetilde{g}$ and $\widetilde{P}_{\mu\nu}(\widetilde{g}, g_b)$ contains only crossed terms between $\widetilde{g}$ and g_b.

4 Outline of the proof of Theorem 1.2

The proof is based on a bootstrap argument. We take $T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$ such that there exists a solution $(\phi, g = g_b + \widetilde{g})$ on $[0,T]$ such that

- $\partial Z^I \phi$ and $\partial Z^I \widetilde{g}$ satisfy weighted L^2 estimates for $I \leq N$.
- $\partial Z^I \phi$ and $\partial Z^I \widetilde{g}$ satisfy point-wise estimates for $I \leq \frac{N}{2}$:

$$|Z^I g_{\underline{LL}}| \leq \frac{C\varepsilon}{(1+|q|)^{\frac{1}{2}-\rho}}, \quad |Z^I \widetilde{g}| \leq \frac{C\varepsilon}{(1+t+r)^{\frac{1}{2}-\rho}},$$

$$|Z^I \phi| \leq \frac{C\varepsilon}{\sqrt{1+t+r}(1+|q|)^{\frac{1}{2}-4\rho}}.$$

- $b(\theta)$ satisfy the estimate

$$\left\| b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q \phi)^2 r dr \right\|_{H^{N-4}(\mathbb{S}^1)} \leq \frac{C\varepsilon^2}{\sqrt{T}}. \tag{20}$$

To prove Theorem 1.2, we show that we can improve these estimates.

4.1 L^2 estimates

To improve the L^2 estimates, we use a weighted energy inequality. We consider a weight function $w(q) \geq 0$ such that $w'(q) \geq 0$.

Proposition 4.1. *We assume that $\Box \phi = f$. Then we have*

$$\frac{1}{2}\partial_t \int w(q)\left((\partial_t \phi)^2 + |\nabla \phi|^2\right) + \frac{1}{2}\int w'(q)\left((\partial_s \phi)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right) \lesssim \int w(q)|f \partial_t \phi|.$$

The use of weights serves many purposes

- They allow to use some Hardy inequalities in order to estimate the weighted L^2 norm of g in term of the weighted L^2 norm of ∂g (with a bigger weight).
- The term in $w'(q)$ in the left-hand-side gives an extra integrability condition for the good derivatives. It allows to make use of the null structure when we estimate $\partial Z^N g$. This is the principle of Alinhac ghost weights, used in [1], and also in [21].
- The decomposition of the metric in the null frame does not commute with the wave operator. It creates terms of the form $\frac{1}{r}\bar{\partial} g$. The use of weights permits to trade a growth in $\sqrt{t}$ in the energy estimates of the bad coefficients (namely $g_{\underline{LL}}$) into a loss in the weight for the good coefficients.

4.2 L^∞ estimates

We can obtain L^∞ estimates from the energy estimates thanks to the weighted Klainerman-Sobolev inequality. The following proposition, proven in the Appendix of [13], concerns the 2+1 dimensional case and is the analogous of Proposition 14.1 in [21] for the $3+1$ dimensional case

Proposition 4.2. *We have the inequality*

$$|f(t,x)w^{\frac{1}{2}}(|x|-t)| \lesssim \frac{1}{\sqrt{1+t+|x|}\sqrt{1+||x|-t|}} \sum_{|I|\leq 2} \|w^{\frac{1}{2}}(.-t)Z^I f\|_{L^2}.$$

With the condition $w'(q) > 0$ for the energy inequality, we are not allowed to take weights of the form $(1+|q|)^{\alpha}$, with $\alpha > 0$ in the region $q < 0$. Therefore, Klainerman-Sobolev inequality cannot give us more than the estimate

$$|\partial u| \lesssim \frac{1}{\sqrt{1+|q|}\sqrt{1+s}},$$

in the region $q < 0$, for a solution of $\Box u = f$. However, we know that for suitable initial data, the solution of the wave equation $\Box u = 0$ satisfies

$$|u| \lesssim \frac{1}{\sqrt{1+|q|}\sqrt{1+s}}, \; |\partial u| \lesssim \frac{1}{(1+|q|)^{\frac{3}{2}}\sqrt{1+s}}.$$

To recover some of this decay we will use the following proposition.

Proposition 4.3. *Let u be a solution of*

$$\begin{cases} \Box u = F, \\ (u, \partial_t u)|_{t=0} = (0,0). \end{cases}$$

For $\mu > \frac{3}{2}, \nu > 1$ we have the following $L^\infty - L^\infty$ estimate

$$|u(t,x)|(1+t+|x|)^{\frac{1}{2}} \leq C(\mu,\nu) M_{\mu,\nu}(F)(1+|t-|x|||)^{-\frac{1}{2}+[2-\mu]_+},$$

where

$$M_{\mu,\nu}(F) = \sup (1+|y|+s)^{\mu}(1+|s-|y||)^{\nu} F(y,s),$$

and where we used the convention $A^{[\alpha]_+} = A^{\max(\alpha,0)}$ if $\alpha \neq 0$ and $A^{[0]_+} = \ln(A)$.

This is proven in the appendix of [13]. This inequality has been introduced by Kubo and Kubota in [17].

4.3 Estimation of $g_{\underline{LL}}$: transport equation

With our new decomposition $g = g_b + \widetilde{g}$, the model problem for $g_{\underline{LL}}$ becomes

$$\begin{cases} \Box \phi = 0, \\ \Box g_{\underline{LL}} = -4(\partial_q \phi)^2 - 4b(\theta)\frac{\partial_q^2(q\chi(q))}{r}. \end{cases} \tag{21}$$

In this paragraph, we focus on this model problem. The analysis can be well adapted to the quasilinear case. To obtain some decay for $g_{\underline{LL}}$, we will approximate it by the solution h_0 of the following transport equation

$$\partial_q h_0 = -4r(\partial_q \phi)^2 - 4b(\theta)\partial_q^2(q\chi(q)). \tag{22}$$

We recall the estimates satisfied by ϕ, solution of the free wave equation with regular enough initial data of size ε :

$$\|\partial \phi\|_{L^2} \leq \varepsilon, \quad |\partial \phi| \leq \frac{\varepsilon}{\sqrt{1+t+r}(1+|q|)^{\frac{3}{2}}}.$$

Consequently (22) yields $\partial_q h_0 = O\left(\frac{\varepsilon^2}{(1+|q|)^3}\right)$. Moreover, we may express the d'Alembertian in coordinates s, q, θ

$$\Box = 4\partial_s\partial_q + \frac{1}{r}(\partial_s + \partial_q) + \frac{1}{r^2}\partial_\theta^2. \tag{23}$$

Therefore, using the fact that ϕ satisfy $\Box\phi = 0$ we obtain

$$\partial_s\partial_q\phi \sim \partial\phi\bar{\partial}\phi = O\left(\frac{\varepsilon^2}{(1+s)^2(1+|q|)^2}\right).$$

Consequently, by integrating this estimate with respect to q we obtain

$$\partial_s h_0 = O\left(\frac{\varepsilon^2}{(1+s)^2}\right). \tag{24}$$

To estimate h_0 itself, we need to use the condition on b, to compensate the integral of $(\partial\phi)^2$ which would lead to an estimate $h_0 = O(\varepsilon^2)$ in the region $q < 0$. If $b(\theta) = \int_{\Sigma_{T,\theta}} (\partial_q\phi)^2 r dr$, by integrating the transport equation (22) at $t = T$, and then (24) at fixed q we obtain

$$h_0 = O\left(\frac{\varepsilon^2}{(1+|q|)}\right). \tag{25}$$

The same estimate is also satisfied by $\partial_\theta h_0$ and $\partial_\theta^2 h_0$. Consequently, thanks to (23) we may calculate

$$\Box h_0 = \frac{\partial_q h_0}{r} + O\left(\frac{\varepsilon^2}{(1+|q|)(1+s)^2}\right) = -4(\partial_q\phi)^2 - 4b(\theta)\frac{\partial_q^2(q\chi(q))}{r} + O\left(\frac{\varepsilon^2}{(1+|q|)(1+s)^2}\right)$$

and thanks to Proposition 4.3, we obtain

$$g_{\underline{LL}} = h_0 + O\left(\frac{\varepsilon^2}{(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{1}{2}-\rho}}\right),$$

which is better that what we would have obtained by applying directly Proposition 4.3 to (21).

By taking into consideration the quasilinear terms, we are led to a loss in $\sqrt{1+|q|}$ in our estimates, which yields the pointwise estimate of Theorem 1.2.

4.4 Improvement of the estimates for $b(\theta)$ and restriction to exponential time

In this paragraph we briefly explain how to improve the estimates for b. The procedure is different for the three first coefficient in the Fourier extension

$$\int b(\theta)d\theta, \quad \int b(\theta)\cos(\theta)d\theta, \int b(\theta)\sin(\theta)d\theta, \tag{26}$$

and for the remaining. The coefficients (26) can not be chosen arbitrarily. They correspond to the asymptotic deficit angle and linear momentum and are imposed by the resolution of the constraint equations (see [12]). We can obtain an estimate of the type of (20) by integrating the constraint equations on the hypersurface $t = T$. The improvement for the remaining is done with an iteration process. The restriction to exponential time comes from the following fact : we have a small growth in the higher energy norms :

$$\|\partial Z^N\phi\|_{L^2} \lesssim \varepsilon(1+t)^{C\sqrt{\varepsilon}}.$$

Consequently, if we set $b(\theta) = \int_{\Sigma_{T,\theta}} (\partial_q\phi)^2 r dr$ we obtain

$$\|\partial^N b(\theta)\|_{L^2} \leq \varepsilon^2(1+T)^{C\sqrt{\varepsilon}},$$

which lead us to assume $T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$.

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