LIMIT EQUATION FOR VACUUM EINSTEIN CONSTRAINTS WITH A TRANSLATIONAL KILLING VECTOR FIELD IN THE COMPACT HYPERBOLIC CASE

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ABSTRACT. We construct solutions to the constraint equations in general relativity using the limit equation criterion introduced in [4]. We focus on solutions over compact 3-manifolds admitting a $\mathbb{S}^1$-symmetry group. When the quotient manifold has genus greater than 2, we obtain strong far from CMC results.

CONTENTS

1. INTRODUCTION

General relativity describes the universe as a $(3+1)$-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric $\mathbf{g}$. The Einstein equations describe how non-gravitational fields influence the curvature of $\mathbf{g}$:

$$\mathbf{Ric}_{\mu\nu} - \frac{\mathbf{Scal}}{2}\mathbf{g}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu},$$

where $\mathbf{Ric}$ and $\mathbf{Scal}$ are respectively the Ricci tensor and the scalar curvature of the metric $\mathbf{g}$ and $\mathbf{T}_{\mu\nu}$ is the sum of the energy-momentum tensors of all the non-gravitational fields.

Einstein equations can be formulated as a Cauchy problem with initial data given by a set $(M, \widehat{g}, \widehat{K})$, where M is a 3-dimensional manifold, $\widehat{g}$ is a Riemannian metric on M and $\widehat{K}$ is a symmetric 2-tensor on M. $\widehat{g}$ and $\widehat{K}$ correspond to the first and second fundamental forms of M seen as an embedded space-like hypersurface in the universe $(\mathcal{M}, \mathbf{g})$ solving the Einstein equations.

It turns out that the Einstein equations imply compatibility conditions on $\widehat{g}$ and $\widehat{K}$ known as the constraint equations:

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$$
\begin{cases}
\mathrm{Scal}_{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} = 2\rho, & (1.1a)\\
\mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K}) = j, & (1.1b)
\end{cases}
$$

where, denoting by N the unit future-pointing normal to M in $\mathcal{M}$, one has

$$\rho = 8\pi \mathbf{T}_{\mu\nu} N^\mu N^\nu, \quad j_i = 8\pi \mathbf{T}_{i\mu} N^\mu.$$

We assume here that μ and ν go from 0 to 3 and denote spacetime coordinates while Latin indices go from 1 to 3 and correspond to coordinates on M.

In this article, to keep things simple, we will consider no field but the gravitational one (vacuum case). As a consequence, we impose $\mathbf{T} \equiv 0$. We will also assume that the spacetime possesses a $\mathbb{S}^1$-symmetry generated by a spacelike Killing vector field. This allows for a reduction of the $(3+1)$-dimensional study of the Einstein equations to a $(2+1)$-dimensional problem. This symmetry assumption has been introduced and studied by Y. Choquet-Bruhat and V. Moncrief in [3] (see also [2]) in the case of a spacetime of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact 2-dimensional manifold of genus $G \geq 2$, $\mathbb{S}^1$ corresponds to the orbit of the $\mathbb{S}^1$-action and $\mathbb{R}$ is the time axis. They proved the existence of global solutions corresponding to perturbations of a particular expanding spacetime. In [3], they use solutions of the constraint equations with constant mean curvature (CMC, i.e. constant $\mathrm{tr}_{\widehat{g}} \widehat{K}$) on the spacelike hypersurface $\Sigma \times \mathbb{S}^1 \times \{0\}$ as initial data. The construction of such solutions is fairly direct. In this article we shall generalize their construction to more general initial data allowing for non-constant mean curvature.

The method which is generally used to construct initial data for the Einstein equations is the conformal method which consists in decomposing the metric $\widehat{g}$ and the second fundamental form $\widehat{K}$ into given data and unknowns that have to be adjusted so that $\widehat{g}$ and $\widehat{K}$ solve the constraint equations, see Section 2. The equations for the unknowns, namely a positive function playing the role of a conformal factor and a 1-form, are usually called the conformal constraint equations. Extended discussion of the conformal method can be found in a series of very nice articles by D. Maxwell [13, 16–18].

These equations have been extensively studied in the case of constant mean curvature (CMC) since the system greatly simplifies in this case. We refer the reader to the excellent review article [1] for an overview of known results in this particular case. The non-CMC case remained open for a couple of decades. Only the case of nearly constant mean curvature was studied. We refer for example to the pioneer work [12]. Two major breakthroughs were obtained in [11], [15] and [4] concerning the far from CMC case. A comparison of these methods is given in [8].

In this article, we follow the method described in [4]. Namely, we give the following criterion: if a certain limit equation admits no non-zero solution, the conformal constraint equations admit at least one solution. The other method [11, 15] would require that Σ is $\mathbb{S}^2$ so that it carries a metric with positive scalar curvature and has no conformal Killing vector field, which is impossible.

This approach has been generalized to the asymptotically hyperbolic case in [9] and to the asymptotically cylindrical case in [6]. The asymptotically Euclidean case [5] and the case of compact manifolds with boundary [7] are currently work in progress since new ideas have to be found to get the criterion.

The outline of the paper is as follows. In Section 2, we show how the Einstein equations reduce to a $(2+1)$-dimensional problem in the case of a $\mathbb{S}^1$-symmetry and exhibit the analog of the conformal constraint equations in this case. We also state Theorem 2.1 which is the main result of this article and Corollary 2.3 which gives an example of application of Theorem 2.1. Section 3 is devoted to the proof of Theorem 2.1. Finally, Section 4 contains the proof of Corollary 2.3.

2. Preliminaries

2.1. **Reduction of the Einstein equations.** Before discussing the constraint equations, we briefly recall the form of the Einstein equations in the presence of a spacelike translational Killing vector field. We follow here the exposition in [2, Section XVI.3].

We recall that we want to write the Einstein equations on the manifold $\mathcal{M} = \Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a Riemannian surface and $\mathbb{R}$ denotes the time direction, for some metric $\mathbf{g}$ which is invariant under translation along the $\mathbb{S}^1$-direction. We let x^3 denote the coordinate along the $\mathbb{S}^1$- direction (seen as $\mathbb{R}/\mathbb{Z}$), choose local coordinates x^1, x^2 on Σ and denote by x^0 the time coordinate.

A metric $\mathbf{g}$ on $\mathcal{M}$ admitting ∂_3 as a Killing vector field has the form

$$\mathbf{g} = \widetilde{g} + e^{2\gamma}\left(dx^3 + A\right)^2,$$

where $\widetilde{g}$ is a Lorentzian metric on $\Sigma \times \mathbb{R}$, A is a 1-form on $\Sigma \times \mathbb{R}$ and γ is a function on $\Sigma \times \mathbb{R}$. Since ∂_3 is a Killing vector field, $\widetilde{g}$, A and γ do not depend on x^3. We set $F = dA$ the field strength of A. The Ricci tensor $\mathbf{Ric}$ of $\mathbf{g}$ can be computed in terms of $\widetilde{g}$, A and γ. In the basis $(dx^0, dx^1, dx^2, dx^3 + A)$, the vacuum Einstein equations ($\mathbf{Ric} = 0$) become

$$\begin{cases} 0 = \mathbf{Ric}_{\alpha\beta} = \widetilde{\mathrm{Ric}}_{\alpha\beta} - \dfrac{1}{2}e^{2\gamma}F_\alpha{}^\lambda F_{\beta\lambda} - \widetilde{\nabla}^2_{\alpha,\beta}\gamma - \nabla_\alpha\gamma\nabla_\beta\gamma, & \text{(2.1a)}\\ 0 = \mathbf{Ric}_{\alpha 3} = \dfrac{1}{2}e^{-\gamma}\widetilde{\nabla}_\beta\left(e^{3\gamma}F_\alpha{}^\beta\right), & \text{(2.1b)}\\ 0 = \mathbf{Ric}_{33} = -e^{-2\gamma}\left(-\dfrac{1}{4}e^{2\gamma}F_{\alpha\beta}F^{\alpha\beta} + \widetilde{g}^{\alpha\beta}\nabla_\alpha\gamma\nabla_\beta\gamma + \widetilde{g}^{\alpha\beta}\widetilde{\nabla}^2_{\alpha,\beta}\gamma\right), & \text{(2.1c)} \end{cases}$$

where the indices α, β and λ go from 0 to 2, and are raised with respect to the metric $\widetilde{g}$. The equation (2.1b) is equivalent to $d(*e^{3\gamma}F) = 0$. So we are going to assume that $*e^{3\gamma}F$ is an exact 1-form. Therefore, there exists a potential $\omega : \Sigma \times \mathbb{R} \to \mathbb{R}$ such that $e^{3\gamma}F = d\omega$.

Defining $\overline{g} = e^{2\gamma}\widetilde{g}$, we obtain the following system for $\overline{g}$, γ and ω:

$$\begin{cases} \Box_{\overline{g}}\omega - 4\overline{\nabla}^\alpha\gamma\overline{\nabla}_\alpha\omega = 0, & \text{(2.2a)}\\ \Box_{\overline{g}}\gamma - \dfrac{1}{2}e^{-4\gamma}\overline{\nabla}^\alpha\omega\overline{\nabla}_\alpha\omega = 0, & \text{(2.2b)}\\ \overline{\mathrm{Ric}}_{\alpha\beta} - 2\overline{\nabla}_\alpha\gamma\overline{\nabla}_\beta\gamma - \dfrac{1}{2}e^{-4\gamma}\overline{\nabla}_\alpha\omega\overline{\nabla}_\beta\omega = 0, & \text{(2.2c)} \end{cases}$$

where $\Box_{\overline{g}} = \overline{g}^{\alpha\beta}\overline{\nabla}^2_{\alpha,\beta}$ is the d'Alembertian associated to the metric $\overline{g}$, $\overline{\mathrm{Ric}}$ is its Ricci tensor and the indices are raised with respect to $\overline{g}$. We introduce the following notation

$$u := (\gamma, \omega),$$

together with the scalar product

$$\partial_\alpha u \cdot \partial_\beta u := 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega.$$

We are going to consider the Cauchy problem for the system (2.2). As for the general Einstein equations, the initial data for this system have to satisfy some constraint equations.

2.2. **The constraint equations.** We write the metric $\overline{g}$ under the following form:

$$\overline{g} = -N^2dt^2 + g_{ij}\left(dx^i + \beta^i dt\right)\left(dx^i + \beta^j dt\right)$$

The coefficient N is called the lapse, while the vector β is called the shift. g is the Riemannian metric induced by $\overline{g}$ on the slices of constant t. We consider the initial data for the spacelike surface Σ which is the constant $t = 0$ hypersurface of $\Sigma \times \mathbb{R}$. We also use the notation

$$\partial_t = \partial_0 - \mathcal{L}_\beta,$$

where $\mathcal{L}_\beta$ is the Lie derivative associated to the vector field β. With this notation, the second fundamental form of $\Sigma \subset \Sigma \times \mathbb{R}$ reads

$$K_{ij} = -\frac{1}{2N}\partial_t g_{ij}.$$

We denote by τ the mean curvature of Σ:

$$\tau := g^{ij}K_{ij}.$$

The constraint equations are obtained by taking the $\partial_t - \partial_t$ and the $\partial_t - \partial_i$ components of the Einstein equations:

$$\begin{cases} \overline{\mathrm{Ric}}_{ti} - \dfrac{\overline{\mathrm{Scal}}}{2}\overline{g}_{ti} = N\left(\partial_i\tau - D^i K_{ij}\right) = \partial_t u \cdot \partial_i u, & \text{(2.3a)} \\ \overline{\mathrm{Ric}}_{tt} - \dfrac{\overline{\mathrm{Scal}}}{2}\overline{g}_{tt} = \dfrac{N^2}{2}\left(\mathrm{Scal} - |K|^2 + \tau^2\right) = \partial_t u \cdot \partial_t u + \dfrac{N^2}{2}\overline{g}^{\alpha\beta}\partial_\alpha u \cdot \partial_\beta u, & \text{(2.3b)} \\ & \text{(2.3c)} \end{cases}$$

where Scal is the scalar curvature of the metric g and D is its Levi-Civita connection. Equation (2.3a) is called the *momentum constraint* while Equation (2.3b) is known as the *Hamiltonian constraint*.

2.3. **The conformal method.** In order to construct solutions to the system (2.3), we are going to use the well-known conformal method which we explain now.

Given a Riemann surface Σ of genus $G \geq 2$, we let g_0 be a metric on Σ with constant scalar curvature $\mathrm{Scal}_0 \equiv -1$ and look for a metric g in the conformal class of g_0:

$$g = e^{2\varphi}g_0$$

for some function $\varphi : \Sigma \to \mathbb{R}$. We also decompose K into a pure trace part and a traceless part,

$$K_{ij} = \frac{\tau}{2}g_{ij} + H_{ij},$$

and, following [3], we set

$$\dot{u} := \frac{e^{2u}}{N}\partial_t u.$$

The system (2.3) then becomes

$$\begin{cases} \nabla^i H_{ij} = -\dot{u} \cdot \partial_j u + \dfrac{e^{2\varphi}}{2}\partial_j\tau, & \text{(2.4a)} \\ \Delta\varphi + e^{-2\varphi}\left(\dfrac{1}{2}\dot{u}^2 + \dfrac{1}{2}|H|^2\right) = e^{2\varphi}\dfrac{\tau^2}{4} - \dfrac{1}{2}\left(1 + |\nabla u|^2\right), & \text{(2.4b)} \end{cases}$$

where ∇ denotes the Levi-Civita connection of the metric g_0, Δ is the Laplace-Beltrami operator of g_0 and from now on, unless stated otherwise, all norms are taken with respect to the metric g_0.

In order to solve Equation (2.4a), we split H according to the York decomposition (see Proposition 3.2 for more details):

$$H = \sigma + LW,$$

where σ is a transverse traceless (TT) tensor, i.e. $\mathrm{tr}_{g_0}\sigma \equiv 0$ and $\nabla^i\sigma_{ij} \equiv 0$, and LW denotes the conformal Killing operator acting on a 1-form W:

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \nabla^k W_k g_{0ij}.$$

The system (2.4) finally becomes

$$\begin{cases} -\frac{1}{2}L^*LW = -\dot{u}\cdot du + \frac{e^{2\varphi}}{2}d\tau, & \text{(2.5a)}\\ \Delta\varphi + e^{-2\varphi}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}\left|\sigma + LW\right|^2\right) = e^{2\varphi}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + \left|\nabla u\right|^2\right), & \text{(2.5b)}\end{cases}$$

where L^* is the formal L^2-adjoint of L:

$$-\frac{1}{2}L^*LW_j = \nabla^i LW_{ij}.$$

The equations of this system are commonly known as the conformal constraint equations. Equation (2.5a) is called the *vector equation* and Equation (2.5b) is named the *Lichnerowicz equation*.

Given u, $\dot{u}$, τ and σ we are going to construct solutions to the system (2.5) for the unknowns φ and W without any smallness assumption on τ. We follow the approach of [4]. The main theorem we prove is the following:

Theorem 2.1. *Given $\dot{u} \in C^0(\Sigma, \mathbb{R})$, $u \in C^1(\Sigma, \mathbb{R})$ $\tau \in W^{1,p}(\Sigma, \mathbb{R})$ and $\sigma \in W^{1,p}$ a TT-tensor, where $p > 2$, and assuming that τ vanishes nowhere on Σ, then at least one of the following assertions is true:*

1. *The set of solutions (φ, W) to the system (2.5) is non-empty and compact in $W^{2,p}(\Sigma, \mathbb{R}) \times W^{2,p}(\Sigma, T^*\Sigma)$*
2. *There exists a non-trivial solution $V \in W^{2,p}(\Sigma, T^*\Sigma)$ of the following limit equation*

$$-\frac{1}{2}L^*LW = \alpha\frac{\sqrt{2}}{2}\left|LW\right|\frac{d\tau}{|\tau|} \tag{2.6}$$

for some $\alpha \in [0, 1]$.

Remark 2.2. Since the surface Σ is of genus $G \geq 2$, there is no conformal Killing vector fields on Σ. Therefore $LW \equiv 0$ imply $W \equiv 0$. In particular, there cannot be any non-zero solution to (2.6) with $\alpha = 0$, since in this case we would have

$$0 = \int_\Sigma \left\langle W, -\frac{1}{2}L^*LW\right\rangle d\mu^{g_0} = -\frac{1}{2}\int_\Sigma \left|LW\right|^2 d\mu^{g_0},$$

which immediately implies that W is a conformal Killing vector field.

The proof of this theorem is the subject of Section 3.

Corollary 2.3. *Assume that the mean curvature τ is such that*

$$\left\|\frac{d\tau}{\tau}\right\|_{L^\infty(\Sigma, T^*\Sigma)} < 1$$

then there exists a solution to the conformal constraint equations (2.4).

See Section 4 for the proof of this corollary.

3. Proof of Theorem 2.1

Before tackling the full system of equations in Subsection 3.3, we first study the properties of each equation individually, in Subsection 3.1 for the vector equation and in Subsection 3.2 for the Lichnerowicz equation.

3.1. **The vector equation.** The main result about Equation (2.4a) is the following:

Proposition 3.1. *Given a 1-form $Y \in L^p(\Sigma, T^*\Sigma)$, there exists a unique $W \in W^{2,p}(\Sigma, T^*\Sigma)$ such that*

$$-\frac{1}{2}L^*LW = Y.$$

Moreover, W satisfies

$$\|W\|_{W^{2,p}(\Sigma,T^*\Sigma)} \lesssim \|Y\|_{L^p(\Sigma,T^*\Sigma)}.$$

Proof. We can write

$$\begin{aligned}
-\frac{1}{2}L^*LW_j &= \nabla^i\left(\nabla_i W_j + \nabla_j W_i - \nabla^k W_k g_{0ij}\right)\\
&= \Delta W_j + \nabla^i\nabla_j W_i - \nabla_j\nabla^i W_i\\
&= \Delta W_j + \mathrm{Ric}_{ij} W^i
\end{aligned}$$

$$-\frac{1}{2}L^*LW_j = \Delta W_j - \frac{1}{2}W_j, \tag{3.1}$$

where we used the fact that in dimension 2, $\mathrm{Ric} = \frac{\mathrm{Scal}}{2} g_{0ij}$. This Bochner formula will be useful in Section 4.

On $W^{1,2}(\Sigma, T^*\Sigma)$, we introduce the following bilinear form

$$a(V,W) := \int_\Sigma \langle LV, LW\rangle\, d\mu^{g_0}.$$

We have

$$\begin{aligned}
a(V,W) &= \int_\Sigma \langle V, L^*LW\rangle\, d\mu^{g_0}\\
&= -2\int_\Sigma \left\langle V, \Delta W - \frac{1}{2}W\right\rangle d\mu^{g_0}\\
&= \int_\Sigma \left(2\langle \nabla V, \nabla W\rangle + \langle V, W\rangle\right) d\mu^{g_0}
\end{aligned}$$

It follows immediately that the bilinear form a satisfies the assumptions of the Lax-Milgram theorem: it is continuous and coercive. So given $Y \in L^p(\Sigma, T^*\Sigma) \subset \left(W^{1,2}(\Sigma, T^*\Sigma)\right)^*$ there exists a unique $W \in W^{1,2}(\Sigma, T^*\Sigma)$ such that $-\frac{1}{2}L^*LW = Y$. It follows from elliptic regularity that $W \in W^{2,p}(\Sigma, T^*\Sigma)$ and that $\|W\|_{W^{2,p}(\Sigma,T^*\Sigma)} \lesssim \|Y\|_{L^p(\Sigma,T^*\Sigma)}$. $\square$

In particular, we get the following result:

Proposition 3.2. *Given a symmetric traceless tensor $H \in W^{1,p}$, there exist a unique TT-tensor σ and a unique 1-form W such that*

$$H = \sigma + LW.$$

Proof. From the previous proposition, there exists a unique solution $W \in W^{2,p}$ of

$$-\frac{1}{2}L^*LW = \mathrm{div}_{g_0} H.$$

Setting $\sigma = H - LW$, we have

$$\mathrm{div}_{g_0}\sigma = \mathrm{div}_{g_0} H - \mathrm{div}_{g_0} LW = \mathrm{div}_{g_0} H + \frac{1}{2}L^*LW = 0.$$

Therefore, σ is a TT-tensor. $\square$

3.2. **The Lichnerowicz equation.** The aim of this section is to prove the following proposition :

Proposition 3.3. *Let $\dot{u}$, u and τ be given as in Theorem 2.1. For any given symmetric traceless 2-tensor $H \in L^\infty$, there exists a unique positive function $\varphi \in W^{2,p}(\Sigma, \mathbb{R})$ solving Equation (2.4b). Further φ depends continuously on $H \in C^0$ and is bounded from below by a positive constant φ_0 which is independent of H.*

Before proving the proposition, we need to recall a general lemma on semilinear elliptic equations. This is a simple version of the so-called sub and super-solution method we took from [20, Chapter 14].

Lemma 3.4. *Given an open interval $I \subset \mathbb{R}$, we consider the following equation for φ on Σ:*

$$\Delta\varphi = f(x, \varphi, \lambda), \tag{3.2}$$

where $\lambda \in \Lambda$ is a parameter belonging to Λ, an open subset of Banach space, and f is a function belonging to $C^0(\Sigma, \mathbb{R}) \otimes C^1(I \times \Lambda, \mathbb{R})$, i.e. f decomposes as a finite sum

$$f = \sum_i a_i(x) f_i(\varphi, \lambda),$$

where $a_i \in C^0(\Sigma, \mathbb{R})$ and $f_i \in C^1(I \times \Lambda, \mathbb{R})$. We assume further that

- $\frac{\partial f}{\partial \varphi} > 0$,
- *there exist constants $a_0, a_1 \in I$ (that may depend continuously on λ), $a_0 \leq a_1$, such that, for all $x \in \Sigma$, $f(x, a_0, \lambda) < 0$ and $f(x, a_1, \lambda) > 0$.*

Then the equation (3.2) admits a unique solution $\varphi \in W^{2,p}(\Sigma, \mathbb{R})$, $2 < p < \infty$, for all $\lambda \in \Lambda$. Further, φ depends continuously on λ.

Proof. We first prove the existence of a solution for all $\lambda \in \Lambda$. We denote by Ω the closed subset of $C^0(M, \mathbb{R})$ defined by

$$\Omega = \{\varphi \in C^0(M, \mathbb{R}), a_0 \leq \varphi \leq a_1\}.$$

We choose a constant $A = A(\lambda) > 0$ such that

$$A > \sup_{(x,\varphi) \in \Sigma \times [a_0, a_1]} \frac{\partial f}{\partial \varphi}(x, \varphi, \lambda)$$

and define a map $F : \Omega \to C^0(M, \mathbb{R})$ as follows. Given $\varphi_0 \in \Omega$, we define $F(\varphi_0) := \varphi_1$, where $\varphi_1 \in W^{2,p}(\Sigma, \mathbb{R})$ is the (unique) solution to the following linear equation:

$$-\Delta\varphi_1 + A\varphi_1 = A\varphi_0 - f(x, \varphi_0, \lambda).$$

We argue that $\varphi_1 \in \Omega$ as follows. We have

$$\begin{aligned}
-\Delta\varphi_1 + A\varphi_1 &= A\varphi_0(x) - f(x, \varphi_0, \lambda) \\
&= \int_{a_0}^{\varphi_0(x)} \underbrace{\left(A - \frac{\partial f}{\partial \varphi}(x, \varphi, \lambda)\right)}_{>0} d\varphi + Aa_0 - f(x, a_0, \lambda) \\
&\geq Aa_0 - f(x, a_0, \lambda) \\
&\geq Aa_0;
\end{aligned}$$

$-\Delta\left(\varphi_1 - a_0\right) + A\left(\varphi_1(x) - a_0\right) \geq 0.$

We set $(\varphi_1 - a_0)_- := \min\{0, \varphi_1 - a_0\}$. Multiplying the previous inequality by $(\varphi_1 - a_0)_-$ and integrating over Σ, we get

$$\int_\Sigma \left[-(\varphi_1 - a_0)_- \Delta\left(\varphi_1 - a_0\right) + A\left(\varphi_1(x) - a_0\right)_-^2\right] d\mu^g \leq 0,$$

$$\int_\Sigma \left[\left|\nabla(\varphi_1 - a_0)_-\right|^2 + A\left(\varphi_1(x) - a_0\right)_-^2\right] d\mu^g \leq 0,$$

from which we immediately conclude that $(\varphi_1(x) - a_0)_- \equiv 0$, i.e. that $\varphi_1 \geq a_0$. A similar argument proves that $\varphi_1 \leq a_1$. Hence F maps Ω into itself.

We note that for fixed λ, F maps Ω into a bounded subset of $W^{2,p}(\Sigma, \mathbb{R})$. This comes from the fact that $\Sigma \times [a_0, a_1]$ is a compact set over which $f(\cdot, \cdot, \lambda)$ is continuous so $f(x, \varphi, \lambda)$ is bounded independently of $\varphi \in \Omega$ and $x \in \Sigma$. Hence, by elliptic regularity

$$\begin{aligned}\|F(\varphi)\|_{W^{2,p}(\Sigma,\mathbb{R})} &\lesssim \|f(x, \varphi, \lambda)\|_{L^\infty(\Sigma,\mathbb{R})}\\ &\lesssim 1.\end{aligned}$$

Denoting by Ω' the closure of the convex hull of $F(\Omega)$, it follows from the Rellich theorem that Ω' is a compact convex subset of $C^0(\Sigma, \mathbb{R})$. By the Schauder fixed point theorem, F admits a fixed point φ. The function φ then satisfies

$$\begin{aligned}-\Delta\varphi + A\varphi &= A\varphi - f(x, \varphi, \lambda)\\ \Leftrightarrow \Delta\varphi &= f(x, \varphi, \lambda).\end{aligned}$$

Hence φ is a solution to (3.2) and by elliptic regularity, $\varphi \in W^{2,p}(\Sigma, \mathbb{R})$.

We next prove that the solution to (3.2) is unique given $\lambda \in \Lambda$. It follows then that $a_0 \leq \varphi \leq a_1$. Assume given φ_1, φ_2 two solutions to (3.2). We have

$$\begin{aligned}0 &= -\Delta(\varphi_2 - \varphi_1) + f(x, \varphi_2, \lambda) - f(x, \varphi_1, \lambda)\\ &= -\Delta(\varphi_2 - \varphi_1) + (\varphi_2 - \varphi_1) \underbrace{\int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_1 + y(\varphi_2 - \varphi_1))dy}_{>0},\end{aligned}$$

from which we immediately conclude that $\varphi_1 \equiv \varphi_2$.

We follow a similar strategy to prove that φ depends continuously on λ. We fix an arbitrary $\lambda_0 \in \Lambda$. There exists $\alpha > 0$ such that

$$\frac{\partial f}{\partial \varphi}(x, \varphi, \lambda_0) \geq \alpha$$

for all $(x, \varphi) \in \Sigma \times [a_0(\lambda_0), a_1(\lambda_0)]$. There exist an $\eta_0 > 0$ and $a'_0, a'_1 \in I$ such that $B_{\eta_0}(\lambda_0) \subset \Lambda$, $a'_0 \leq a_0(\lambda)$, $a'_1 \geq a_1(\lambda)$ for all $\lambda \in B_{\eta_0}(\lambda_0)$ and

$$\frac{\partial f}{\partial \varphi}(x, \varphi, \lambda) > \frac{\alpha}{2}$$

on $\Sigma \times [a'_0, a'_1] \times B_{\eta_0}(\lambda_0)$. We denote by φ_0 the solution to (3.2) with $\lambda = \lambda_0$.

For any $\epsilon > 0$, there exists $\eta > 0$, $\eta < \eta_0$ such that

$$|f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0)| < \frac{\epsilon\alpha}{2}$$

for all $x \in \Sigma$ and all $\lambda \in B_\eta(\lambda_0)$. We denote by φ_1 the solution to (3.2) with $\lambda = \lambda_1$ for an arbitrary $\lambda_1 \in B_\eta(\lambda_0)$:

$$\begin{cases}-\Delta\varphi_0 + f(x, \varphi_0, \lambda_0) = 0\\ -\Delta\varphi_1 + f(x, \varphi_1, \lambda_1) = 0\end{cases}$$

Subtracting both equations, we get

$$\begin{aligned}0 &= -\Delta(\varphi_1 - \varphi_0) + f(x, \varphi_1, \lambda_1) - f(x, \varphi_0, \lambda_0)\\ &= -\Delta(\varphi_1 - \varphi_0) + f(x, \varphi_1, \lambda_1) - f(x, \varphi_0, \lambda_1) + f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0)\end{aligned}$$

(3.3)

$$0 = -\Delta(\varphi_1 - \varphi_0) + \int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1)dy(\varphi_1 - \varphi_0) + f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0).$$

From our assumptions, we have

$$\int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1) dy > \frac{\alpha}{2}.$$

Multiplying Equation (3.3) by $(\varphi_1 - \varphi_0 - \epsilon)_+ := \max\{0, \varphi_1 - \varphi_0 - \epsilon\} \geq 0$, and integrating over Σ, we get

$$\begin{aligned}
&\int_\Sigma \left(f(x, \varphi_0, \lambda_0) - f(x, \varphi_0, \lambda_1)\right)(\varphi_1 - \varphi_0 - \epsilon)_+ d\mu^{g_0} \\
&\qquad = \int_\Sigma \Big[\langle \nabla(\varphi_1 - \varphi_0 - \epsilon)_+, \nabla(\varphi_1 - \varphi_0 - \epsilon)_+ \rangle \\
&\qquad\qquad + \int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1) dy (\varphi_1 - \varphi_0)(\varphi_1 - \varphi_0 - \epsilon)_+ \Big] d\mu^{g_0}, \\
&\int_\Sigma \frac{\epsilon\alpha}{2}(\varphi_1 - \varphi_0 - \epsilon)_+ d\mu^{g_0} \\
&\qquad \geq \int_\Sigma \left[|\nabla(\varphi_1 - \varphi_0 - \epsilon)_+|^2 + \frac{\alpha}{2}(\varphi_1 - \varphi_0)(\varphi_1 - \varphi_0 - \epsilon)_+ \right] d\mu^{g_0} \\
&0 \geq \int_\Sigma \left[|\nabla(\varphi_1 - \varphi_0 - \epsilon)_+|^2 + \frac{\alpha}{2}\left((\varphi_1 - \varphi_0 - \epsilon)_+\right)^2 \right] d\mu^{g_0}
\end{aligned}$$

Hence $\varphi_1 - \varphi_0 \leq \epsilon$. Similarly, $\varphi_1 - \varphi_0 \geq -\epsilon$. This proves that the function Ψ mapping λ to φ solving (3.2) is continuous from Λ to $C^0(\Sigma, I)$. It then follows at once from elliptic regularity that Ψ is continuous as a mapping from Λ to $W^{2,p}(\Sigma, \mathbb{R})$. □

We refer the reader to [14, Section 6] for much stronger versions of the sub and super-solution method. We can now give the proof of Proposition 3.3:

Proof of Proposition 3.3. The Lichnerowicz equation (2.4b) can be rewritten in the form (3.2):

$$\Delta\varphi = \underbrace{-e^{-2\varphi}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) + e^{2\varphi}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right)}_{:=f(x,\varphi)}.$$

Since τ^2 is bounded away from zero, the assumption $\frac{\partial f}{\partial \varphi} > 0$ is readily checked. Choosing $a_0 := -\max \ln|\tau|$, we have

$$e^{2a_0}\frac{\tau^2}{4} \leq \frac{1}{4}.$$

So

$$f(x, a_0) \leq e^{2a_0}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right) \leq \frac{1}{4} - \frac{1}{2} \leq -\frac{1}{4}.$$

Since f is increasing with φ, we immediately get that if $\varphi < a_0$, then $f(x, \varphi) < 0$. Since τ^2 is bounded away from zero we can found $a_1 \geq 0$ be such that

$$e^{2a_1}\frac{\min\tau^2}{4} > \frac{1}{2}\left(1 + \|\nabla u\|^2_{L^\infty}\right) + \frac{1}{2}\|\dot{u}\|^2_{L^\infty} + \frac{1}{2}\|H\|^2_{L^\infty}.$$

Using the fact that we choose $a_1 \geq 0$, it is a simple matter to check that

$$f(x, a_1) > 0$$

and hence if $\varphi > a_1$, $f(x, \varphi) > 0$.

As a consequence, the Lichnerowicz equation satisfies the assumptions of Lemma 3.4. This completes the proof of Proposition 3.3. □

3.3. **The coupled system.** Following [19], we use Schaefer's fixed point theorem to study the coupled system (see [10, Chapter 11]):

Theorem 3.5. *Let X be a Banach space and $\Phi : X \to X$ a continuous compact mapping. Assume that the set*

$$F := \{x \in X, \exists \rho \in [0,1], x = \rho\Phi(x)\}$$

is bounded. Then Φ has a fixed point:

$$\exists x \in X, x = \Phi(x),$$

and the set of fixed points is compact.

We choose $X = C^0(\Sigma, \mathbb{R})$ as a Banach space and construct the mapping Φ as follows: Given $v \in X$,

- From Proposition 3.1 there exists a unique $W := W(v) \in W^{2,p}$ solving
$$-\frac{1}{2}L^*LW = -\dot{u}\cdot du + \frac{v^2}{2}d\tau, \tag{3.4}$$
which is Equation 2.5a with $e^\varphi = v$. Further W depends continuously on $v \in C^0$ for the $W^{2,p}$-norm.
- $W \in W^{2,p}$ can then be continuously mapped to $H := \sigma + LW \in W^{1,p}$
- and, in turn, H can be compactly embedded into C^0.
- Proposition 3.3 yields a unique $\varphi \in W^{2,p}$ solving the Lichnerowicz equation (2.4b) with the H we previously found.

Setting $\Phi(v) := e^\varphi \in C^0(\Sigma, \mathbb{R})$, we loop the loop providing a continuous compact map $\Phi : X \to X$. Thus, we are almost under the assumptions of Theorem 3.5. All we need to check is that the set F is bounded. This is the content of the next proposition:

Proposition 3.6. *Assume that the set*

$$F := \{v \in L^\infty(\Sigma, \mathbb{R}), \exists \rho \in [0,1], v = \rho\Phi(v)\}$$

is unbounded. Then there exists a constant $\rho_0 \in [0,1]$ and a non-zero $W \in W^{2,p}$ such that

$$-\frac{1}{2}L^*LW = \frac{\sqrt{2}}{2}\rho_0 \left|LW\right| \frac{d\tau}{|\tau|}.$$

Proof. Assuming that F is unbounded, we can find sequences $(\rho_i)_{i\geq 0}$ and $(v_i)_{i\geq 0}$ such that $0 \leq \rho_i \leq 1$, $v_i = \rho_i\Phi(v_i)$ and $\|v_i\|_{L^\infty} \to \infty$. Setting $\varphi_i = \log(\Phi(v_i))$ (i.e. $v_i = \rho_i e^{\varphi_i}$), and defining W_i as the solution to (3.4) with $v \equiv v_i$, we get the following equations:

$$\begin{cases} -\frac{1}{2}L^*LW_i = -\dot{u}\cdot du + \rho_i^2\frac{e^{2\varphi_i}}{2}d\tau, & \text{(3.5a)}\\ \Delta\varphi_i + e^{-2\varphi_i}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}\left|\sigma + LW_i\right|^2\right) = e^{2\varphi_i}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right), & \text{(3.5b)}\end{cases}$$

Following [4, 9, 19], we set $\gamma_i := \|e^{\varphi_i}\|_{L^\infty}$ and we introduce the following rescaled objects:

$$\psi_i := \varphi_i - \log(\gamma_i),\ \widetilde{W}_i := \frac{1}{\gamma_i^2}W_i.$$

Note that since we assumed that $\|v_i\|_{L^\infty} = \rho_i\gamma_i \to \infty$, with $0 \leq \rho_i \leq 1$, we also have that $\gamma_i \to \infty$. We will assume without loss of generality that $\gamma_i \geq 1$. The following equations for ψ_i and $\widetilde{W}_i$ follow from the definition:

$$\begin{cases} -\frac{1}{2}L^*L\widetilde{W}_i = -\frac{1}{\gamma_i^2}\dot{u}\cdot du + \rho_i^2\frac{e^{2\psi_i}}{2}d\tau, & \text{(3.6a)}\\ \frac{1}{\gamma_i^2}\Delta\psi_i + e^{-2\psi_i}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) = e^{2\psi_i}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}\left(1+|\nabla u|^2\right), & \text{(3.6b)}\end{cases}$$

Note that $\left\|e^{\psi_i}\right\|_{L^\infty} = \left\|\frac{1}{\gamma_i}e^{\varphi_i}\right\|_{L^\infty} = 1$. Hence, from Proposition 3.1 applied to (3.6a), we have

$$\begin{aligned}\left\|\widetilde{W}_i\right\|_{W^{2,p}} &\lesssim \left\|-\frac{1}{\gamma_i^2}\dot{u}\cdot du + \rho_i^2\frac{e^{2\psi_i}}{2}d\tau\right\|_{L^p}\\ &\lesssim \frac{1}{\gamma_i^2}\left\|\dot{u}\cdot du\right\|_{L^p} + \left\|d\tau\right\|_{L^p}\\ &\lesssim 1.\end{aligned}$$

Consequently, $\widetilde{W}_i$ is bounded in $W^{2,p}$. Since the embedding $W^{2,p}\hookrightarrow C^1$ is compact, we can assume, up to extraction of a subsequence, that $\widetilde{W}_i$ converges to some $\widetilde{W}_\infty \in W^{2,p}$ for the C^1-norm. We can also assume that $\rho_i \to \rho_\infty \in [0,1]$. All we need to do is to prove that $e^{2\psi_i}$ converges in L^∞ to $f_\infty := \sqrt{2}\frac{|L\widetilde{W}_\infty|}{|\tau|}$.

Indeed, passing to the limit in Equation (3.6a), we get that $\widetilde{W}_\infty$ satisfies

$$\begin{aligned}-\frac{1}{2}L^*L\widetilde{W}_\infty &= \rho_\infty^2\frac{f_\infty}{2}d\tau\\ &= \frac{\sqrt{2}}{2}\rho_\infty^2\left|L\widetilde{W}_\infty\right|\frac{d\tau}{|\tau|}.\end{aligned} \tag{3.7}$$

Hence, $\widetilde{W}_\infty$ satisfies the limit equation with $\alpha = \rho_\infty^2$. Since $e^{2\psi_i}$ has L^∞-norm 1 and converges in L^∞ to f_∞, we have $\|f_\infty\|_{L^\infty} = 1$. In particular, $L\widetilde{W}_\infty \not\equiv 0$ which proves that $\widetilde{W}_\infty \not\equiv 0$.

To prove convergence of $e^{2\psi_i}$ to f_∞, we show that for any $\epsilon > 0$, there exists an i_0 such that

$$\left|e^{2\psi_i} - f_\infty\right| \leq \epsilon$$

for all $i \geq i_0$. We do it in two steps:

- We first show the upper bound

$$e^{2\psi_i} \leq f_\infty + \epsilon$$

by selecting a smooth function f_+ such that

$$f_\infty + \frac{\epsilon}{2} \leq f_+ \leq f_\infty + \epsilon$$

and proving that for i_0 large enough, $\psi_+ := \frac{1}{2}\log(f_+)$ is a super-solution to (3.6b):

$$\frac{1}{\gamma_i^2}\Delta\psi_+ + e^{-2\psi_+}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) \leq e^{2\psi_+}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}\left(1+|\nabla u|^2\right). \tag{3.8}$$

Since $f_\infty \geq 0$, $f_+ \geq \frac{\epsilon}{2}$ so ψ_+ is a smooth function. In particular, $|\Delta \psi_+|$ is bounded. Moreover, since $\widetilde{W}_i \to \widetilde{W}_\infty$ in C^1 and $\gamma_i \to \infty$, we have

$$\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2 \to \left|L\widetilde{W}_\infty\right|^2$$

as i tends to infinity. So the condition (3.8) can be rephrased as

$$o(1) + \frac{1}{2}\left|L\widetilde{W}_\infty\right|^2 - \frac{\tau^2}{4} f_+^2 \leq 0,$$

where $o(1)$ denotes a sequence of functions tending uniformly to 0 when $i \to \infty$. We have

$$f_+^2 \geq \left(f_\infty + \frac{\epsilon}{2}\right)^2 \geq f_\infty^2 + \frac{\epsilon^2}{4}.$$

This yields, for i big enough,

$$o(1) + \frac{1}{2}\left|L\widetilde{W}_\infty\right|^2 - \frac{\tau^2}{4} f_+^2 \leq o(1) + \frac{\tau^2}{4} f_\infty^2 - \frac{\tau^2}{4}\left(f_\infty^2 + \frac{\epsilon^2}{4}\right) \leq o(1) - \frac{\tau_0^2 \epsilon^2}{4} \leq 0,$$

where $\tau_0^2 := \inf_\Sigma \tau^2$ is positive by assumption. Therefore ψ_+ is a super-solution to Equation (3.6b) and we obtain, for i big enough

$$\begin{aligned}
\frac{1}{\gamma_i^2}\Delta(\psi_+ - \psi_i) &\leq -\left(e^{-2\psi_+} - e^{-2\psi_i}\right)\left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right) + \frac{\tau^2}{4}\left(e^{2\psi_+} - e^{2\psi_i}\right) \\
&\leq \frac{\tau^2}{2} e^{2\psi_i}(\psi_+ - \psi_i)\int_0^1 e^{2\lambda(\psi_+ - \psi_i)} d\lambda \\
&\quad + \left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right) e^{-2\psi_i}(\psi_+ - \psi_i)\int_0^1 e^{-2\lambda(\psi_+ - \psi_i)} d\lambda \\
&\leq \underbrace{\left[\frac{\tau^2}{2} e^{2\psi_i}\int_0^1 e^{2\lambda(\psi_+ - \psi_i)} d\lambda + \left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right) e^{-2\psi_i}\int_0^1 e^{-2\lambda(\psi_+ - \psi_i)} d\lambda\right]}_{>0}(\psi_+ - \psi_i).
\end{aligned}$$

The maximum principle implies that $\psi_i \leq \psi_+$, for i big enough, so

$$e^{2\psi_i} \leq f_\infty + \epsilon.$$

• Second, we show the lower bound

$$e^{2\psi_i} \geq f_\infty - \epsilon$$

We have to be more careful than for the super-solution, since f_∞ can vanish somewhere. Let $\mathcal{A}$ be a smooth open domain such that

$$\left\{f_\infty^2 \geq \frac{3\epsilon^2}{4}\right\} \subset \mathcal{A} \subset \left\{f_\infty^2 \geq \frac{2\epsilon^2}{3}\right\}.$$

Denote by r the distance to $\partial\mathcal{A}$. On $\mathcal{A}$ we have

$$f_\infty^2 - \frac{\epsilon^2}{2} \geq \frac{2\epsilon^2}{3} - \frac{\epsilon^2}{2} \geq \frac{\epsilon^2}{6}.$$

Let f_- be a smooth function on $\mathcal{A}$ such that

- we have the inequality, $f_\infty^2 - \epsilon^2 \leq f_-^2 \leq f_\infty^2 - \frac{\epsilon^2}{2}$,
- $f_- > 0$,
- in a neighbourhood of $\partial\mathcal{A}$ we have $f_- = \epsilon e^{-\frac{1}{r}}$. This is compatible with the first point since

$$f_\infty^2 - \epsilon^2 \leq \frac{3\epsilon^2}{4} - \epsilon^2 \leq -\frac{\epsilon^2}{4} \text{ on } \partial\mathcal{A}.$$

On $\mathcal{A}$, we can define $\psi_- = \frac{1}{2}\ln(f_-)$. We want to show that the following inequality is satisfied on $\mathcal{A}$:

$$\frac{1}{\gamma_i^2}\Delta\psi_- + e^{-2\psi_-}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) \geq e^{2\psi_-}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}(1+|\nabla u|^2). \quad (3.9)$$

Since $e^{2\psi_-} > 0$ on $\mathcal{A}$, this is equivalent to showing

$$\frac{1}{\gamma_i^2}e^{2\psi_-}\left(\Delta\psi_- + \frac{1}{2}(1+|\nabla u|^2)\right) + \left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) - e^{4\psi_-}\frac{\tau^2}{4} \geq 0.$$

We calculate

$$e^{2\psi_-}\Delta\psi_- = \frac{1}{2}\left[\Delta f_- - \frac{|\nabla f_-|^2}{f_-}\right].$$

Since $f_- \equiv \epsilon e^{-1/r}$ in a neighbourhood of $\partial\mathcal{A}$, we see that $e^{2\psi_-}\Delta\psi_-$ is bounded on $\mathcal{A}$. Therefore, as for the upper bound, the condition (3.9) can be written

$$o(1) + \frac{1}{2}|LW_\infty|^2 - e^{4\psi_-}\frac{\tau^2}{4} \geq 0.$$

On $\mathcal{A}$ we have $e^{4\psi_-} \leq f_-^2 \leq f_\infty^2 - \frac{\epsilon}{2}$. This yields, for i big enough,

$$o(1) + \frac{1}{2}|LW_\infty|^2 - e^{4\psi_-}\frac{\tau^2}{4} \geq o(1) + \frac{\tau^2}{4}f_\infty^2 - \frac{\tau^2}{4}\left(f_\infty^2 - \frac{\epsilon}{2}\right) \geq o(1) + \frac{\tau^2}{4}\frac{\epsilon}{2} \geq 0.$$

Since $\psi_-(x) - \psi_i(x) \to -\infty$ when $x \to \partial\mathcal{A}$, $\psi_-(x) - \psi_i(x)$ attains its maximum on $\mathcal{A}$. Therefore, since ψ_- is a subsolution, we can apply the maximum principle on $\mathcal{A}$, to deduce that $\psi_- \leq \psi_i$. This yields on $\mathcal{A}$

$$f_\infty^2 - \epsilon^2 \leq e^{4\psi_i}.$$

On the complement of $\mathcal{A}$ we have

$$f_\infty^2 - \epsilon^2 \leq 0 \leq e^{4\psi_i}.$$

This concludes the proof of the convergence in L^∞ of $e^{2\psi_i}$ towards f_∞. □

4. Proof of Corollary 2.3

To prove Corollary 2.3, all we need to do is to prove that the limit equation (2.6) admits no non-zero solution under the assumption

$$\left\|\frac{d\tau}{\tau}\right\|_{L^\infty(\Sigma, T^*\Sigma)} < 1.$$

We take the scalar product of the limit equation with W and integrate over Σ. From the Bochner formula (3.1), we get:

$$-\frac{1}{2}\int_\Sigma |LW|^2\, d\mu^{g_0} = \alpha\frac{\sqrt{2}}{2}\int_\Sigma |LW|\left\langle W, \frac{d\tau}{|\tau|}\right\rangle d\mu^{g_0}$$

$$\int_\Sigma |\nabla W|^2\, d\mu^{g_0} + \frac{1}{2}\int_\Sigma |W|^2\, d\mu^{g_0} \leq \alpha\sqrt{2}\int_\Sigma |\nabla W|\left|\frac{d\tau}{\tau}\right||W| d\mu^{g_0}$$

$$\leq \alpha\int_\Sigma |\nabla W|^2\, d\mu^{g_0} + \frac{\alpha}{2}\int_\Sigma\left|\frac{d\tau}{\tau}\right|^2 |W|^2 d\mu^{g_0}$$

$$\frac{1}{2}\int_\Sigma |W|^2\, d\mu^{g_0} \leq \frac{\alpha}{2}\left\|\frac{d\tau}{\tau}\right\|_{L^\infty}^2 \int_\Sigma |W|^2 d\mu^{g_0},$$

where we used the well-known inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \sqrt{2}\,|\nabla W|$ and $b = \left|\frac{d\tau}{\tau}\right| |W|$. The last inequality immediately yields that $W \equiv 0$ since we assumed that $\left\|\frac{d\tau}{\tau}\right\|^2_{L^\infty} < 1$ and $\alpha \in [0, 1]$.

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